12.6 Homogeneous Functions

A function is said to be homogeneous of degree r, if multiplication of each of its independent variables by a constant j will alter the value of the function by the proportion j^r, that is, if

$$f(jx_1, \ldots, jx_n) = j^r f(x_1, \ldots, x_n)$$

In general, j can take any value. However, in order for the preceding equation to make sense, $(jx_1, \ldots, jx_n)$ must not lie outside the domain of the function f. For this reason, in economic applications the constant j is usually taken to be positive, as most economic variables do not admit negative values.

Example 1

Given the function $f(x, y, w) = x/y + 2w/3x$, if we multiply each variable by j, we get

$$f(jx, jy, jw) = \frac{(jx)}{(jy)} + \frac{2(jw)}{3(jx)} = \frac{x}{y} + \frac{2w}{3x} = f(x, y, w) = j^0 f(x, y, w)$$

In this particular example, the value of the function will *not* be affected at all by equal proportionate changes in all the independent variables; or, one might say, the value of the function is changed by a multiple of j^0 $(= 1)$. This makes the function f a homogeneous function of degree zero.

You will observe that the functions x^* and y^* cited at the end of Sec. 12.5 are both homogeneous of degree zero.

Example 2

When we multiply each variable in the function

$$g(x, y, w) = \frac{x^2}{y} + \frac{2w^2}{x}$$

by j, we get

$$g(jx, jy, jw) = \frac{(jx)^2}{(jy)} + \frac{2(jw)^2}{(jx)} = j\left(\frac{x^2}{y} + \frac{2w^2}{x}\right) = jg(x, y, w)$$

The function g is homogeneous of degree one (or, of the first degree); multiplication of each variable by j will alter the value of the function exactly j-fold as well.

Example 3

Now, consider the function $h(x, y, w) = 2x^2 + 3yw - w^2$. A similar multiplication this time will give us

$$h(jx, jy, jw) = 2(jx)^2 + 3(jy)(jw) - (jw)^2 = j^2 h(x, y, w)$$

Thus the function h is homogeneous of degree two; in this case, a doubling of all variables, for example, will quadruple the value of the function.

Linear Homogeneity

In the discussion of production functions, wide use is made of homogeneous functions of the first degree. These are often referred to as *linearly homogeneous* functions, the adverb *linearly* modifying the adjective *homogeneous*. Some writers, however, seem to prefer the somewhat misleading terminology *linear* homogeneous functions, or even *linear and* homogeneous functions, which tends to convey, wrongly, the impression that the functions themselves are linear. On the basis of the function g in Example 2, we know that a function which is homogeneous of the first degree is *not necessarily* linear in itself. Hence you should avoid using the terms "linear homogeneous functions" and "linear and homogeneous functions" unless, of course, the functions in question are indeed linear. Note, however, that it is not incorrect to speak of "linear homogeneity," meaning homogeneity of degree one, because to modify a noun (homogeneity) does call for the use of an adjective (linear).

Since the primary field of application of linearly homogeneous functions is in the theory of production, let us adopt as the framework of our discussion a production function in the form, say,

$$Q = f(K, L) \tag{12.45}$$

Whether applied at the *micro* or the *macro* level, the mathematical assumption of linear homogeneity would amount to the economic assumption of constant returns to scale, because linear homogeneity means that raising all inputs (independent variables) j-fold will always raise the output (value of the function) exactly j-fold also.

What unique properties characterize this linearly homogeneous production function?

Property I Given the linearly homogeneous production function $Q = f(K, L)$, the average physical product of labor (APP_L) and of capital (APP_K) can be expressed as functions of the capital–labor ratio, $k \equiv K/L$, alone.

To prove this, we multiply each independent variable in (12.45) by a factor $j = 1/L$. By virtue of linear homogeneity, this will change the output from Q to $jQ = Q/L$. The right side of (12.45) will correspondingly become

$$f\left(\frac{K}{L}, \frac{L}{L}\right) = f\left(\frac{K}{L}, 1\right) = f(k, 1)$$

Since the variables K and L in the original function are to be replaced (whenever they appear) by k and 1, respectively, the right side in effect becomes a function of the capital–labor ratio k alone, say, $\phi(k)$, which is a function with a single argument, k, even though two independent variables K and L are actually involved in that argument. Equating the two sides, we have

$$\text{APP}_L \equiv \frac{Q}{L} = \phi(k) \tag{12.46}$$

The expression for APP_K is then found to be

$$\text{APP}_K \equiv \frac{Q}{K} = \frac{Q}{L}\frac{L}{K} = \frac{\phi(k)}{k} \tag{12.47}$$

Since both average products depend on k alone, linear homogeneity implies that, as long as the K/L ratio is kept constant (whatever the absolute levels of K and L), the average products will be constant, too. Therefore, while the production function is homogeneous of degree one, both APP_L and APP_K are homogeneous of degree *zero* in the variables K and L, since equal proportionate changes in K and L (maintaining a constant k) will not alter the magnitudes of the average products.

Property II Given a linearly homogeneous production function $Q = f(K, L)$, the marginal physical products MPP_L and MPP_K can be expressed as functions of k alone.

To find the marginal products, we first write the total product as

$$Q = L\phi(k) \qquad \text{[by (12.46)]} \tag{12.45'}$$

and then differentiate Q with respect to K and L. For this purpose, we shall find the following two preliminary results to be of service:

$$\frac{\partial k}{\partial K} = \frac{\partial}{\partial K}\left(\frac{K}{L}\right) = \frac{1}{L} \qquad \frac{\partial k}{\partial L} = \frac{\partial}{\partial L}\left(\frac{K}{L}\right) = \frac{-K}{L^2} \tag{12.48}$$

The results of differentiation are

$$\begin{aligned}
\text{MPP}_K \equiv \frac{\partial Q}{\partial K} &= \frac{\partial}{\partial K}[L\phi(k)] \\
&= L\frac{\partial \phi(k)}{\partial K} = L\frac{d\phi(k)}{dk}\frac{\partial k}{\partial K} && \text{[chain rule]} \\
&= L\phi'(k)\left(\frac{1}{L}\right) = \phi'(k) && \text{[by(12.48)]} \qquad (12.49)
\end{aligned}$$

$$\begin{aligned}
\text{MPP}_L \equiv \frac{\partial Q}{\partial L} &= \frac{\partial}{\partial L}[L\phi(k)] \\
&= \phi(k) + L\frac{\partial \phi(k)}{\partial L} && \text{[product rule]} \\
&= \phi(k) + L\phi'(k)\frac{\partial k}{\partial L} && \text{[chain rule]} \\
&= \phi(k) + L\phi'(k)\frac{-K}{L^2} && \text{[by (12.48)]} \\
&= \phi(k) - k\phi'(k) && (12.50)
\end{aligned}$$

which indeed show that MPP_K and MPP_L are functions of k alone.

Like average products, the marginal products will remain the same as long as the capital-labor ratio is held constant; they are homogeneous of degree zero in the variables K and L.

Property III (Euler's theorem) If $Q = f(K, L)$ is linearly homogeneous, then

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} \equiv Q$$

PROOF

$$\begin{aligned}
K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} &= K\phi'(k) + L[\phi(k) - k\phi'(k)] && \text{[by (12.49), (12.50)]} \\
&= K\phi'(k) + L\phi(k) - K\phi'(k) && [k \equiv K/L] \\
&= L\phi(k) = Q && \text{[by (12.45')]}
\end{aligned}$$

Note that this result is valid for *any* values of K and L; this is why the property can be written as an identical equality. What this property says is that the value of a linearly homogeneous function can always be expressed as a sum of terms, each of which is the product of one of the independent variables and the first-order partial derivative with respect to that variable, regardless of the levels of the two inputs actually employed. Be careful, however, to distinguish between the identity $K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} \equiv Q$ [Euler's theorem, which applies only to the constant-returns-to-scale case of $Q = f(K, L)$] and the equation $dQ = \frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL$ [total differential of Q, for *any* function $Q = f(K, L)$].

Economically, this property means that under conditions of constant returns to scale, if each input factor is paid the amount of its marginal product, the total product will be exactly exhausted by the distributive shares for all the input factors, or, equivalently, the pure economic profit will be zero. Since this situation is descriptive of the long-run equilibrium under pure competition, it was once thought that only linearly homogeneous production functions would make sense in economics. This, of course, is not the case. The zero economic profit in the long-run equilibrium is brought about by the forces of competition through the entry and exit of firms, regardless of the specific nature of the production functions actually prevailing. Thus it is not mandatory to have a production function that ensures product exhaustion for any and all (K, L) pairs. Moreover, when imperfect competition exists in the factor markets, the remuneration to the factors may not be equal to the marginal products, and, consequently, Euler's theorem becomes irrelevant to the distribution picture. However, linearly homogeneous production functions are often convenient to work with because of the various nice mathematical properties they are known to possess.

Cobb-Douglas Production Function

One specific production function widely used in economic analysis (earlier cited in Sec. 11.6, Example 5) is the *Cobb-Douglas production function:*

$$Q = AK^{\alpha}L^{1-\alpha} \tag{12.51}$$

where A is a positive constant, and α is a positive fraction. What we shall consider here first is a generalized version of this function, namely,

$$Q = AK^{\alpha}L^{\beta} \tag{12.52}$$

where β is another positive fraction which may or may not be equal to $1 - \alpha$. Some of the major features of this function are: (1) it is homogeneous of degree $(\alpha + \beta)$; (2) in the special case of $\alpha + \beta = 1$, it is linearly homogeneous; (3) its isoquants are negatively sloped throughout and strictly convex for positive values of K and L; and (4) it is strictly quasiconcave for positive K and L.

Its homogeneity is easily seen from the fact that, by changing K and L to jK and jL, respectively, the output will be changed to

$$A(jK)^{\alpha}(jL)^{\beta} = j^{\alpha+\beta}(AK^{\alpha}L^{\beta}) = j^{\alpha+\beta}Q$$

That is, the function is homogeneous of degree $(\alpha + \beta)$. In case $\alpha + \beta = 1$, there will be constant returns to scale, because the function will be linearly homogeneous. (Note, however, that this function is *not* linear! It would thus be confusing to refer to it as a "*linear* homogeneous" or "*linear* and homogeneous" function.) That its isoquants have negative slopes and strict convexity can be verified from the signs of the derivatives dK/dL and

d^2K/dL^2 (or the signs of dL/dK and d^2L/dK^2). For any positive output Q_0, (12.52) can be written as

$$AK^\alpha L^\beta = Q_0 \qquad (A, K, L, Q_0 > 0)$$

Taking the natural log of both sides and transposing, we find that

$$\ln A + \alpha \ln K + \beta \ln L - \ln Q_0 = 0$$

which implicitly defines K as a function of L.† By the implicit-function rule and the log rule, therefore, we have

$$\frac{dK}{dL} = -\frac{\partial F/\partial L}{\partial F/\partial K} = -\frac{(\beta/L)}{(\alpha/K)} = -\frac{\beta K}{\alpha L} < 0$$

Then it follows that

$$\frac{d^2K}{dL^2} = \frac{d}{dL}\left(-\frac{\beta K}{\alpha L}\right) = -\frac{\beta}{\alpha}\frac{d}{dL}\left(\frac{K}{L}\right) = -\frac{\beta}{\alpha}\frac{1}{L^2}\left(L\frac{dK}{dL} - K\right) > 0$$

The signs of these derivatives establish the isoquant (any isoquant) to be downward-sloping throughout and strictly convex in the LK plane for positive values of K and L. This, of course, is only to be expected from a function that is strictly quasiconcave for positive K and L. For the strict quasiconcavity feature of this function, see Example 5 of Sec. 12.4, where a similar function was discussed.

Let us now examine the $\alpha + \beta = 1$ case (the Cobb-Douglas function proper), to verify the three properties of linear homogeneity cited earlier. First of all, the total product in this special case is expressible as

$$Q = AK^\alpha L^{1-\alpha} = A\left(\frac{K}{L}\right)^\alpha L = LAk^\alpha \tag{12.51'}$$

where the expression Ak^α is a specific version of the general expression $\phi(k)$ used before. Therefore, the average products are

$$\begin{aligned} \text{APP}_L &= \frac{Q}{L} = Ak^\alpha \\ \text{APP}_K &= \frac{Q}{K} = \frac{Q}{L}\frac{L}{K} = \frac{Ak^\alpha}{k} = Ak^{\alpha-1} \end{aligned} \tag{12.53}$$

both of which are now functions of k alone.

Second, differentiation of $Q = AK^\alpha L^{1-\alpha}$ yields the marginal products:

$$\begin{aligned} \frac{\partial Q}{\partial K} &= A\alpha K^{\alpha-1}L^{-(\alpha-1)} = A\alpha\left(\frac{K}{L}\right)^{\alpha-1} = A\alpha k^{\alpha-1} \\ \frac{\partial Q}{\partial L} &= AK^\alpha(1-\alpha)L^{-\alpha} = A(1-\alpha)\left(\frac{K}{L}\right)^\alpha = A(1-\alpha)k^\alpha \end{aligned} \tag{12.54}$$

and these are also functions of k alone.

† The conditions of the implicit-function theorem are satisfied, because F (the left-side expression) has continuous partial derivatives, and because $\partial F/\partial K = \alpha/K \neq 0$ for positive values of K.

Last, we can verify Euler's theorem by using (12.54) as follows:

$$K\frac{\partial Q}{\partial K}+L\frac{\partial Q}{\partial L}=KA\alpha k^{\alpha-1}+LA(1-\alpha)k^{\alpha}$$
$$=LAk^{\alpha}\left(\frac{K\alpha}{Lk}+1-\alpha\right)$$
$$=LAk^{\alpha}(\alpha+1-\alpha)=LAk^{\alpha}=Q \qquad \text{[by (12.51')]}$$

Interesting economic meanings can be assigned to the exponents α and $(1-\alpha)$ in the linearly homogeneous Cobb-Douglas production function. If each input is assumed to be paid by the amount of its marginal product, the relative share of total product accruing to capital will be

$$\frac{K(\partial Q/\partial K)}{Q}=\frac{KA\alpha k^{\alpha-1}}{LAk^{\alpha}}=\alpha$$

Similarly, labor's relative share will be

$$\frac{L(\partial Q/\partial L)}{Q}=\frac{LA(1-\alpha)k^{\alpha}}{LAk^{\alpha}}=1-\alpha$$

Thus the exponent of each input variable indicates the relative share of that input in the total product. Looking at it another way, we can also interpret the exponent of each input variable as the partial elasticity of output with respect to that input. This is because the capital-share expression just given is equivalent to the expression $\frac{\partial Q/\partial K}{Q/K}\equiv\varepsilon_{QK}$ and, similarly, the labor-share expression just given is precisely that of ε_{QL}.

What about the meaning of the constant A? For given values of K and L, the magnitude of A will proportionately affect the level of Q. Hence A may be considered as an *efficiency parameter*, i.e., as an indicator of the state of technology.

Extensions of the Results

We have discussed linear homogeneity in the specific context of production functions, but the properties cited are equally valid in other contexts, provided the variables K, L, and Q are properly reinterpreted.

Furthermore it is possible to extend our results to the case of more than two variables. With a linearly homogeneous function

$$y=f(x_1,x_2,\ldots,x_n)$$

we can again divide each variable by x_1 (that is, multiply by $1/x_1$) and get the result

$$y=x_1\phi\left(\frac{x_2}{x_1},\frac{x_3}{x_1},\ldots,\frac{x_n}{x_1}\right) \qquad \text{[homogeneity of degree 1]}$$

which is comparable to (12.45'). Moreover, Euler's theorem is easily extended to the form

$$\sum_{i=1}^{n}x_i f_i\equiv y \qquad \text{[Euler's theorem]}$$

where the partial derivatives of the original function f (namely, f_i) are again homogeneous of degree zero in the variables x_i, as in the two-variable case.

The preceding extensions can, in fact, also be generalized with relative ease to a homogeneous function of degree r. In the first place, by definition of homogeneity, we can in the present case write

$$y = x_1^r \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}\right) \qquad \text{[homogeneity of degree } r\text{]}$$

The modified version of Euler's theorem will now appear in the form

$$\sum_{i=1}^{n} x_i f_i \equiv ry \qquad \text{[Euler's theorem]}$$

where a multiplicative constant r has been attached to the dependent variable y on the right. And, finally, the partial derivatives of the original function f, the f_i, will all be homogeneous of degree $(r - 1)$ in the variables x_i. You can thus see that the linear-homogeneity case is merely a special case thereof, in which $r = 1$.

EXERCISE 12.6

1. Determine whether the following functions are homogeneous. If so, of what degree?
   (a) $f(x, y) = \sqrt{xy}$
   (b) $f(x, y) = (x^2 - y^2)^{1/2}$
   (c) $f(x, y) = x^3 - xy + y^3$
   (d) $f(x, y) = 2x + y + 3\sqrt{xy}$
   (e) $f(x, y, w) = \dfrac{xy^2}{w} + 2xw$
   (f) $f(x, y, w) = x^4 - 5yw^3$
2. Show that the function (12.45) can be expressed alternatively as $Q = K\psi\left(\frac{L}{K}\right)$ instead of $Q = L\phi\left(\frac{K}{L}\right)$.
3. Deduce from Euler's theorem that, with constant returns to scale:
   (a) When $MPP_K = 0$, APP_L is equal to MPP_L.
   (b) When $MPP_L = 0$, APP_K is equal to MPP_K.
4. On the basis of (12.46) through (12.50), check whether the following are true under conditions of constant returns to scale:
   (a) An APP_L curve can be plotted against k ($= K/L$) as the independent variable (on the horizontal axis).
   (b) MPP_K is measured by the slope of that APP_L curve.
   (c) APP_K is measured by the slope of the radius vector to the APP_L curve.
   (d) $MPP_L = APP_L - k(MPP_K) = APP_L - k$ (slope of APP_L).
5. Use (12.53) and (12.54) to verify that the relations described in Prob. 4*b*, *c*, and *d* are obeyed by the Cobb-Douglas production function.
6. Given the production function $Q = AK^\alpha L^\beta$, show that:
   (a) $\alpha + \beta > 1$ implies increasing returns to scale.
   (b) $\alpha + \beta < 1$ implies decreasing returns to scale.
   (c) α and β are, respectively, the partial elasticities of output with respect to the capital and labor inputs.

7. Let output be a function of three inputs: $Q = AK^aL^bN^c$.
   (a) Is this function homogeneous? If so, of what degree?
   (b) Under what condition would there be constant returns to scale? Increasing returns to scale?
   (c) Find the share of product for input N, if it is paid by the amount of its marginal product.
8. Let the production function $Q = g(K, L)$ be homogeneous of degree 2.
   (a) Write an equation to express the second-degree homogeneity property of this function.
   (b) Find an expression for Q in terms of $\phi(k)$, in the vein of (12.45′).
   (c) Find the MPP_K function. Is MPP_K still a function of k alone, as in the linear-homogeneity case?
   (d) Is the MPP_K function homogeneous in K and L? If so, of what degree?

12.7 Least-Cost Combination of Inputs

As another example of constrained optimization, let us discuss the problem of finding the least-cost input combination for the production of a specified level of output Q_0 representing, say, a customer's special order. Here we shall work with a general production function; later on, however, reference will be made to homogeneous production functions.

First-Order Condition

Assuming a smooth production function with two variable inputs, $Q = Q(a, b)$, where $Q_a, Q_b > 0$, and assuming both input prices to be exogenous (though again omitting the zero subscript), we may formulate the problem as one of minimizing the cost

$$C = aP_a + bP_b$$

subject to the output constraint

$$Q(a, b) = Q_0$$

Hence, the Lagrangian function is

$$Z = aP_a + bP_b + \mu[Q_0 - Q(a, b)]$$

To satisfy the first-order condition for a minimum C, the input levels (the choice variables) must satisfy the following simultaneous equations:

$$Z_\mu = Q_0 - Q(a, b) = 0$$
$$Z_a = P_a - \mu Q_a = 0$$
$$Z_b = P_b - \mu Q_b = 0$$

The first equation in this set is merely the constraint restated, and the last two imply the condition

$$\frac{P_a}{Q_a} = \frac{P_b}{Q_b} = \mu \tag{12.55}$$