Dept. of Mathematics
4^{th} Semester
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Riemann Integration

Assignment 1

① Let $f : [a, b] \to \mathbb{R}$ be bounded and monotone increasing on $[a, b]$. If P_n be the partition of $[a, b]$ dividing into n-sub-intervals of eaual length prove that

$$\int_a^b f \le U(P_n, f) \le \int_a^b f + \frac{b-a}{n}[f(b) - f(a)].$$

② A function f is defined on $[0, 1]$ by $f(x) = x^2 + x^3,\ x \in \mathbb{Q}$
$= x + x^2,\ x \in \mathbb{R} \setminus \mathbb{Q}$

(i) Evaluate $\underline{\int_0^1} f,\ \overline{\int_0^1} f$; (ii) Show that f is not integrable on $[0, 1]$

③ Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Prove that

(i) $\max(f, g) : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$

(ii) $\min(f, g) : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$

④ A function f is defined on $[0, 1]$ by $f(0) = 0$ and

$f(x) = 0$, if x be irrational

$= \frac{1}{q}$, if $x = \frac{p}{q}$, where p, q are positive integers prime to each other

show that f is integrable on $[0, 1]$ and $\int_0^1 f(x)\,dx = 0$.

⑤ Suppose f is a bounded real function on $[a, b]$, and $f^2 \in R[a, b]$. Does it follow that $f \in R[a, b]$?

Does the answer change if we assume that $f^3 \in R[a, b]$

⑥ (i) Let $f(x) = \lim_{n \to \infty} \frac{x^n + 3}{x^n + 1}$, $0 \le x \le 2$

State, with reasons, whether f is Riemann-integrable on $[0, 2]$

(ii) A function $f : [0, 1] \to [0, 1]$ is defined as follows

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \quad (n = 0, 1, 2, \ldots) \\ 0 & \text{if } x = 0 \end{cases}$$

show that f is Riemann integrable over $[0, 1]$

(7) Prove that there exists a point $c \in [0, 1]$ such that

$$\int_0^1 \frac{\sin \pi x}{1+x^2}\, dx = \frac{2}{\pi(1+c^2)}$$

(8) Show that $\dfrac{\pi^3}{24\sqrt{2}} < \displaystyle\int_0^{\pi/2} \frac{x^2}{\sin x + \cos x}\, dx < \dfrac{\pi^3}{24}$

(9) Let f be real-valued and continuous for all $x \geq 0$ and $f(x) \neq 0$ for all $x \geq 0$. If $\{f(x)\}^2 = \int_0^x f(t)\,dt$ prove that $f(x) = x$ for all $x \geq 0$.

(10) prove that $\displaystyle\lim_{x \to 0} \frac{\int_0^{x^2} e^{\sqrt{1+t}}}{x^2} = e$.