2021

MATHEMATICS — HONOURS

Paper : DSE-B(2)-3

(Advanced Mechanics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and Notations have their usual meanings.

Group - A

1. Answer the following multiple choice questions with only one correct option. Choose the correct option and justify : 2×10

(a) A constraint of the form :

$$\vec{f}\left(\vec{r}_j, t\right) = 0,\ j = 1, 2, \ldots, N$$

is called a

(i) kinematical constraint (ii) bilateral constraint

(iii) unilateral constraint (iv) geometric constraint.

(b) The number of degrees of freedom of a rigid body is

(i) 9 (ii) 3

(iii) 6 (iv) None of these.

(c) If the Hamiltonian of a one-dimensional dynamical system is given by

$$H = \frac{1}{2}\left(\alpha p^2 + \beta q^2 + 2\gamma pq\right),$$

where α, β, γ are constants, then the Poisson bracket $\{p, \mathrm{H}\}$ is equal to

(i) $-\alpha q + \gamma p$ (ii) $\beta q + \gamma p$

(iii) $-\alpha q - \gamma p$ (iv) $-\beta q - \gamma p$.

(d) Two weightless, inextensible rods AB and BC are suspended at A and jointed by a flexible joint at B. The degrees of freedom of the system is

(i) 3 (ii) 4

(iii) 5 (iv) 6.

(e) The Lagrangian of a free particle of mass m and velocity v is given by

(i) $L = \frac{mv}{2}$ (ii) $L = -\frac{mv^2}{2}$

(iii) $L = \frac{mv^2}{2}$ (iv) $L = -\frac{mv}{2}$.

(f) The Hamiltonian of a particle of mass m is $H = \frac{p^2}{2m} + pq$, where q is the generalized coordinate and p is the corresponding momentum. The Lagrangian of the particle is

(i) $\frac{m}{2}(\dot{q}+q)^2$ (ii) $\frac{m}{2}(\dot{q}-q)^2$

(iii) $\frac{m}{2}(\dot{q}^2+q\dot{q}-q^2)$ (iv) $\frac{m}{2}(\dot{q}^2-q\dot{q}+q^2)$.

(g) Consider the Hamiltonian (H) and the Lagrangian (L) for a free particle of mass m and velocity v. Then

(i) H and L are independent of each other

(ii) H and L are related but have different dependence on v

(iii) H and L are equal

(iv) H is a quadratic in v but L is not.

(h) 'If a symmetry is found to exist in a dynamical problem, then there is a corresponding constant of motion.'— This is

(i) Noether's theorem (ii) Fermat's principle

(iii) Liouville's theorem (iv) None of these.

(i) A dynamical system having kinetic energy T and potential energy V is described by the Hamiltonian H. Assume that the equations defining the generalized coordinates do not depend on time t. Then,

(i) $H = T + V$ and is conserved

(ii) $H = T + V$ but it is not conserved

(iii) $T + V$ is conserved but is not equal to H

(iv) none of these.

(j) A mechanical system is described by the Hamiltonian $H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$. As a result of canonical transformation generated by $F(q, Q) = -\frac{Q}{q}$, the Hamiltonian in the new coordinate Q and new momentum P becomes

(i) $\frac{1}{2m}Q^2P^2+\frac{1}{2}m\omega^2Q^2$ (ii) $\frac{1}{2m}Q^2P^2+\frac{1}{2}m\omega^2P^2$

(iii) $\frac{1}{2m}P^2+\frac{1}{2}m\omega^2Q^2$ (iv) $\frac{1}{2m}Q^2P^4+\frac{1}{2}m\omega^2P^{-2}$.

Group - B

Unit - 1

2. Answer ***any two*** questions : 5×2

(a) What is degree of freedom? Define holonomic and scleronomic constraints. Write the type of constraint(s) for motion of a body on an inclined plane under gravity. 1+2+2

(b) What do you understand by cyclic coordinates? Show that the generalized momentum corresponding to a cyclic coordinate is a constant of motion. 2+3

(c) A double pendulum consists of a mass m_1 attached to one end a massless rod of length l_1 whose other end is hinged, and a second mass m_2 attached to another massless rod of length l_2 whose other end is hinged to the mass m_1. Using suitable generalized coordinates, setup the Lagrangian and write the Lagrange's equations for the system. 5

(d) What are dissipative forces? What is Rayleigh's dissipation function and how is it related with the rate of energy dissipation of a system? Use Rayleigh's dissipation function to write Lagrange's equations of motion for a dissipative system where the nonpotential forces are linear in the generalized velocities. 1+2+2

Unit - 2

3. Answer ***any three*** questions : 5×3

(a) In a dynamical system, the kinetic and potential energies are $T=\frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2}+\frac{1}{2}\dot{q}_2^2$,

$$V=C+dq_2^2 \text{ where } a,\ b,\ c,\ d \text{ are constants.}$$

Determine $q_1(t)$ and $q_2(t)$ by Routh's process of ignoration of coordinates. 5

(b) A certain oscillator with generalized coordinate q has Lagrangian

$$L=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2.$$

Verify that $q^*=\sin t$ is a motion of the oscillator, and show directly that it makes the action integral stationary in any time interval $[0, \tau]$. 5

(c) What is Legendre transformation? Using Legendre transformation, obtain Hamilton's equation of motion from Lagrange's equation of motion. 2+3

(d) State Hamilton's principle. Derive Hamilton's principle from D'Alembert's principle. 1+4

(e) (i) The Lagrangian of a system with two degrees of freedom is given by $L = \frac{1}{2} m \dot{x}^2 + m \dot{x} \dot{y}$. Write down the Hamiltonian of the system.

(ii) Suppose that the equations defining generalized coordinates q_i for a system with n degrees of freedom do not involve time. Show that in this case $\sum_{i=1}^{n} p_i \dot{q}_i = 2T$, where p_i are the momenta conjugate to q_i and T is the kinetic energy of the system. 3+2

Unit - 3

4. Answer ***any two*** questions : 5×2

(a) Find Hamilton's equations in spherical polar coordinates for a particle of mass m moving in three dimensions in a force field of potential V. 5

(b) Explain the principle of stationary action. How does this principle lead to Fermat's principle? 3+2

(c) A spherical pendulum consists of a mass m attached to one end of a massless rod of length l which is hinged at the other end and is free to oscillate in any vertical plane. Using suitable generalized coordinates, write down the Hamiltonian and find the equations of motion for the coordinates. 5

(d) What are Poincaré integral invariants and Poincaré–Cartan integral invariants? What is the difference between the two? 2+2+1

Unit - 4

5. Answer ***any two*** questions : 5×2

(a) Show that the transformation

$$P = q \cot p$$
$$Q = \log\left(\frac{\sin p}{q}\right)$$

is canonical. Show also that the generating function for this transformation is

$$F = e^{-Q}\left(1 - e^{2Q}\right)^{\frac{1}{2}} + q \sin^{-1} q \, e^{Q}$$ 3+2

(b) Show that the function $S = \frac{m\omega}{2}\left(q^2 + \alpha^2\right)\cot \omega t - m\omega q \alpha \operatorname{cosec} \omega t$ is a solution of the Hamilton–Jacobi equation for Hamilton's principal function for the linear harmonic oscillator with

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right).$$

Show that this function generates a correct solution to the motion of the harmonic oscillator. 3+2

(c) Define the Poisson bracket of two dynamical variables $u = u\,(q_i, p_i, t)$ and $v = v\,(q_i, p_i, t)$ where q_i, p_i are the canonical variables and t is time.

Show that Poisson brackets are canonical invariants. 2+3

(d) State and prove Liouville's theorem for a Hamiltonian system. 1+4

2021

MATHEMATICS — HONOURS

Paper : DSE-B(2)-2

(Astronomy and Space Science)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations have usual meanings.

Group - A

1. Answer all the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) A telescope observing in space at a wavelength of 800 nm has an aperture with a diameter of 5 m. What is its angular resolution?

(i) 1.95×10^{-7} arcsec (ii) 4.03×10^{-2} arcsec

(iii) 1.95×10^{-1} arcsec (iv) 1.6 arcsec.

(b) A star of magnitude +4 lies at a distance of 100 *pc*. Then the absolute magnitude of the star is

(i) + 9.0 (ii) + 4.0

(iii) + 1.49 (iv) – 1.0.

(c) The redshift of a nearby galaxy is 0.01. If the Hubble constant is 73 km s^{-1} Mpc^{-1}, then the distance of the galaxy in Mpc is

(i) 7.3 Mpc (ii) 21.9 Mpc

(iii) 41.1 Mpc (iv) 730 Mpc.

(d) The microwave background radiation has a spectrum which peaks at a wavelength of 1.1 mm and is identical in shape to that of a black body of temperature 2.7 K. At what wavelength will the spectrum of the star Sirius A (with temperature 9940 K) peak?

(i) 9036 nm (ii) 335 nm

(iii) 299 nm (iv) 34 nm.

(e) The sun will spend 1.1×10^{10} yr on the main sequence. Given that the main sequence stars obey a mass luminosity relationship of the form $L \propto M^{3.5}$. What is the lifetime of a $3M_{\odot}$ star? ($M_{\odot}$ represents solar mass)

(i) 1.08×10^{8} yr (ii) 9.05×10^{8} yr

(iii) 2.13×10^{8} yr (iv) 6.9×10^{8} yr.

(f) A star has a parallax of 0.01 *arcseconds*. Then the distance of the star will be

(i) 3.26 light years (ii) 326 light years

(iii) 100 light years (iv) 10 light years.

(g) The distance of the Sun from the centre of our galaxy is 8.5 *kpc*. What will be the circular velocity of the Sun around the galactic centre?

[Take the constants $A = 14.4$ km s^{-1} kpc^{-1} and $B = -12$ km s^{-1} kpc^{-1}]

(i) 250 km s^{-1} (ii) 224.4 km s^{-1}

(iii) 242.2 km s^{-1} (iv) 220.1 km s^{-1}.

(h) Suppose we look at two distant galaxies : Galaxy 1 is twice as far away as Galaxy 2. In that case

(i) We are seeing Galaxy 1 as it looked at an earlier time in the history of the universe than Galaxy 2

(ii) We are seeing Galaxy 1 as it looked at a later time in the history of the universe than Galaxy 2

(iii) Galaxy 1 must be twice as big as Galaxy 2

(iv) Galaxy 2 must be twice as old as Galaxy 1.

(i) The dimensions of the Reynold's number is

(i) $[M^2L^3T]$ (ii) $[ML^3T]$

(iii) $[M^2L^2T^2]$ (iv) None of these.

(j) The expansion of the universe will be halted if the mass density of the Universe be equal to the critical density ρ_c whose value is [Take $H = 70$ km s^{-1} Mpc^{-1}]

(i) 0.5×10^{-29} gm cm^{-3} (ii) 1×10^{-29} gm cm^{-3}

(iii) 1.5×10^{-29} gm cm^{-3} (iv) 2×10^{-29} gm cm^{-3}.

Group - B

2. Answer ***any one*** question : 5×1

(a) In connection with the spherical triangle, given the observer's latitude 'φ', the declination 'δ' and hour angle 'H' of the heavenly body, calculate its zenith distance and azimuth. Also given the observer's latitude 'φ', the star's zenith distance 'z' and azimuth 'A', calculate the star's declination and hour angle. 3+2

(b) Derive the fundamental formula of spherical trigonometry. 5

Group - C

3. Answer ***any one*** question : 5×1

(a) Discuss the different layers of Earth's atmosphere, indicating the major constituents and their interaction with electromagnetic radiation of different wavelengths. 5

(b) What is f/a ratio of a telescope and what are its various advantages? Compare the brightness of images of the Moon produced by two telescopes – one with $f = 200$ cm, $a = 40$ cm, and the other with $f = 600$ cm and $a = 100$ cm. 2+3

Group - D

4. Answer ***any two*** questions : 5×2

(a) Define luminosity of a star. What is its relation with the effective temperature of a star? Derive the relationship between the luminosity and the absolute magnitude of a star. 1+1+3

(b) What is stellar parallax? The apparent magnitude of a star is observed to be +3.3 and its parallax is $0''.025$. Find the absolute magnitude of the star. Compare the luminosity of this star with that of the Sun ($M_{v\odot} = +5.0$). 1+2+2

(c) The coronal spectrum shows emission lines of intense ionization— Explain. Comment on the sources of the coronal heating. 3+2

(d) Discuss the solar neutrino puzzle and its possible solutions. 5

Group - E

5. Answer ***any one*** question : 5×1

(a) What are interstellar shock waves? Write down the equations which are appropriate for studying the propagation of a plane, normal and adiabatic shock. Deduce the Rankine-Hugoniot relation. 1+2+2

(b) Define Jeans wavelength, λ_j and Jeans Mass M_j. How are they related to the gravitational collapse of a static homogeneous cloud? Derive expressions for them. 1+1+3

Group - F

6. Answer ***any two*** questions : 5×2

(a) Derive the formulae for the radial velocity, v_r and the tangential velocity, v_T in terms of the Oort's constants A and B. 5

(b) Draw a diagram of the rotation curve of our galaxy and obtain a polynomial in the radial distance 'r' that fits the rotation curve fairly well. 2+3

(c) Describe Hubble's morphological classification of galaxies. What are the principal observable features that form the basis for this classification? What features distinguish the sub-classes? 2+2+1

(d) Discuss the observations that suggest that a very large fraction of matter remains hidden in individual galaxies, galaxy clusters and in the universe. Also derive an estimate of the hidden matter. 3+2

Group - G

7. Answer ***any two*** questions : 5×2

(a) If 'm_0' and 'm_f' are respectively the initial and final mass of a rocket, then prove that $m_f = m_0 \exp\left(-\frac{\Delta v}{c}\right)$, where Δv is the difference between the initial and final velocity of the rocket and 'c' is the velocity of exhaust. 5

(b) As an approximate of Navier–Stokes equation of motion, derive the boundary layer equations for two-dimensional incompressible fluid flow past a flat plate. 5

(c) What is Blasius boundary layer flow? Deduce the self-similar equation for this flow. 1+4

(d) Write a note on the remarkable achievements of the Indian Space Research Organization (ISRO). 5

[Throughout the Paper take the Newton's Gravitational constant as $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}].

2021

MATHEMATICS — HONOURS

Paper : DSE-B(2)-1

(Point Set Topology)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all the following multiple choice questions. For each question, **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) If τ_1 and τ_2 are the topologies on $\mathbb{R}^2$ generated by the base β_1 of interiors of all circular regions in $\mathbb{R}^2$ and the base β_2 of interiors of all rectangular regions in $\mathbb{R}^2$ respectively, then

(i) τ_1 is a proper subset of τ_2 (ii) τ_2 is a proper subset of τ_1

(iii) $\tau_1 = \tau_2$ (iv) $\tau_1 \cap \tau_2 = \{\mathbb{R}^2, \emptyset\}$.

(b) Let (X, τ) be a topological space and A be a non-empty subset of X such that every non-empty open subset of X intersects A. Then which of the following is true?

(i) A must be equal to X (ii) A is dense in X

(iii) $A = \bar{A}$ (iv) A must be an open set.

(c) Let (X, τ) be a topological space and A be a non-empty proper subset of X such that the boundary of A is an empty set. Then which of the following is false?

(i) A contains all of its limit points

(ii) Every point of A is an interior point

(iii) The boundary of $(X \setminus A)$ is an empty set

(iv) A is closed but may not be an open set.

(d) An uncountable set with cofinite topology is

(i) both T_1 and first countable space.

(ii) both T_2 and first countable space.

(iii) a first countable space but not a T_2 space.

(iv) neither first countable nor a T_2 space.

(e) Let $f:(\mathbb{R},\tau_u)\to(\mathbb{R},\tau_u)$ be a continuous map (where τ_u denotes the usual topology on $\mathbb{R}$) and $Z(f)=\{x\in\mathbb{R}: f(x)=0\}$. Which of the following is true?

(i) $Z(f)$ must be a closed set (ii) $Z(f)$ must be compact

(iii) $Z(f)$ must be an open set (iv) $Z(f)$ must be connected.

(f) The number of T_1 topologies that can be defined on a finite set with n elements is

(i) 1 (ii) n

(iii) 2^n (iv) $n-1$.

(g) Which of the following statements is not correct for the discrete topology τ_d on $\mathbb{R}$?

(i) τ_d is the largest topology on $\mathbb{R}$

(ii) $(\mathbb{R}, \tau_d)$ is compact

(iii) $(\mathbb{R}, \tau_d)$ is first countable

(iv) For every subset A of $\mathbb{R}$, $A^{\circ}=\bar{A}$,

where A° and $\bar{A}$ denotes the interior and closure of A in $(\mathbb{R}, \tau_d)$.

(h) If $\tau=\{\phi,\{a\},\{a,b\},X\}$ is a topology on $X=\{a, b, c\}$, then (X, τ) is

(i) compact and Hausdorff (ii) compact but not Hausdorff

(iii) only Hausdorff (iv) neither compact nor Hausdorff.

(i) Which of the following statements is not true?

(i) $\mathbb{R}$ with usual topology is homeomorphic with the subspace topology on $(-1, 1)$.

(ii) $\left[-1,\frac{1}{2}\right)$ is open in $[-1, 1]$ with respect to the subspace topology from the usual topology on $\mathbb{R}$.

(iii) $[-1, 1]$ is homeomorphic with $\{(x,y)\in\mathbb{R}^2: x^2+y^2=1\}$, where both the sets are endowed with the subspace topology from the usual topology on $\mathbb{R}$ and product topology on $\mathbb{R}^2$ respectively.

(iv) $[-1, 1]$ is homeomorphic with $[0, 1]$, where both the sets are endowed with the subspace topology from the usual topology on $\mathbb{R}$.

(j) Let $X=\mathbb{N}\times\mathbb{Q}$ with the subspace topology of $\mathbb{R}^2$ and $P=\left\{\left(n,\frac{1}{n}\right): n\in\mathbb{N}\right\}$. Which of the following statements is true?

(i) P is closed but not open (ii) P is open but not closed

(iii) P is both open and closed (iv) P is neither open nor closed.

Unit - 1

(Marks : 20)

Answer ***any four*** questions.

2. Let (X, τ) be the topological product of the family of topological spaces $\{(X_i, \tau_i) : i = 1, 2, ..., n\}$ and $p_i : X \to X_i$ denote the i th projection map $\forall i = 1, 2, ..., n$. Prove that

(a) p_i is an open map for each i

(b) τ is the smallest topology on X such that each p_i is continuous. 2+3

3. Prove that a topological invariant is a metric invariant. Is the converse true? Justify. 3+2

4. Let (X, d) be a metric space and A be a nonempty subset of X. Prove that the function $f_A : (X, \tau(d)) \to \mathbb{R}$ defined by $f_A(x) = \inf\, \{d(x, a) : a \in A\}$, $\forall x \in X$ is continuous on X (where $\tau(d)$ denotes the metric topology on X induced by d). Hence prove that for any $A \subseteq X$,

$$\overline{A} = \{x \in X : d(x, A) = 0\} \text{ in } (X, \tau(d))$$ 3+2

5. (a) τ is the usual topology on $\mathbb{R}$ and $\tau' = \{A \cup B : A \in \tau, B \subseteq \mathbb{R} \setminus \mathbb{Q}\}$. Prove that τ' is a topology on $\mathbb{R}$ which is finer than τ.

(b) Find the interior of the set $\{\sqrt{2} + n : n \in \mathbb{N}\}$ in $(\mathbb{R}, \tau')$. 3+2

6. (a) Prove that an isometry $f : (X, d) \to (Y, d')$ is a homeomorphism from $(X, \tau(d))$ to $(Y, \tau(d'))$. (Here (X, d) and (Y, d') are two metric spaces and $\tau(d)$ and $\tau(d')$ are the topologies generated by the corresponding metric on X and Y respectively.)

(b) If $\{A_\alpha : \alpha \in \Lambda\}$ is an infinite family of subsets in any topological space (X, τ), then the equality $\overline{\bigcup_{\alpha \in \Lambda} A_\alpha} = \bigcup_{\alpha \in \Lambda} \overline{A_\alpha}$ is always true— correct or justify. 3+2

7. (X, τ) is a topological space and D is a dense subset of X.

(a) Prove that, for an open subset Y of X, $D \cap Y$ is dense in the subspace topology on Y. Is the result true if Y is not open? Justify.

(b) Prove that for a continuous surjection $f : (X, \tau) \to (Z, \tau')$ the set $f(D)$ is dense in Z, where (Z, τ') is any topological space. 3+2

8. If (X, τ) is a second countable space and B is a base for τ, then prove that there exists a countable subfamily D of B such that D is a base for τ. 5

Unit - 2

(Marks : 10)

Answer ***any two*** questions.

9. Let $f: X \to Y$ be any function from a topological space X into a topological space Y. If f is continuous, then prove that the graph of f defined by $G(f) = \{(X, f(x)) : x \in X\}$ is homeomorphic to X. 5

10. (a) Prove that a topological space (X, τ) is Hausdorff if the diagonal $\{(x, x) : x \in X\}$ is a closed set in the product space $(X \times X,\ \tau \times \tau)$.

(b) Prove or disprove : In a topological space (X, τ), if every covergent sequence in X has unique limit then X is a T_2 space. 3+2

11. (a) $f: (X, \tau) \to (Y, \tau')$ is an open, continuous, surjection and (X, τ) is a first countable space. Prove that Y is first countable.

(b) Consider a topology η on $\mathbb{R}$ given by $\eta = \{U \subseteq \mathbb{R} :$ either $1 \notin U$ or $\mathbb{R} \backslash U$ is finite$\}$. Prove that $(\mathbb{R}, \eta)$ is not first countable. 3+2

12. (a) $f: (X, \tau) \to (Y, \tau')$ is a continuous and injective, where Y is a Hausdorff space. Show that X is Hausdorff.

(b) If (X, τ) is a T_1 space and every intersection of open sets is open in (X, τ), prove that τ is the discrete topology on X. 3+2

Unit - 3

(Marks : 15)

Answer ***any three*** questions.

13. (a) Prove or disprove : The intersection of any family of compact subsets of a space is compact.

(b) Prove or disprove : $(\mathbb{R}, \tau_c)$ is a compact space, where $\tau_c = \{U \subseteq \mathbb{R} :$ either $\mathbb{R} \backslash U$ is countable or $\mathbb{R}\}$ 3+2

14. (a) A and B are two compact subsets of a space (X, τ) such that each point of A is strongly separated from each point of B. Prove that A and B are strongly separated in X.

(b) 'There does not exist a continuous map from [2, 5] onto (1, 4), where [2, 5] and (1, 4) are endowed with the subspace topology of the usual topology on $\mathbb{R}$'— Justify the statement. 3+2

15. (a) In a topological space (X, τ), E is a connected subset of X so that $E = A \cup B \cup C$, where A and B are separated and C is connected. Show that $A \cup C$ is connected.

(b) Consider $\mathbb{R}$ endowed with the usual topology, $f: \mathbb{R} \to \mathbb{R}$ is any function such that $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \backslash \mathbb{Q}) \subseteq \mathbb{Q}$. Show that f is not a continuous function. 3+2

16. (a) If every real valued continuous function defined on a topological space X takes on every value between any two values that it assumes then prove that X is connected.

(b) Prove that a continuous mapping from a connected space to the real line having only rational values is constant. 3+2

17. (a) If A is a connected subset of a metric space (X, d) consisting of atleast two points then prove that A is uncountable.

(b) Find all components of the set of rational numbers endowed with the subspace topology from the usual topology of $\mathbb{R}$. 3+2