T(6th Sm.)-Mathematics-H/CC-13/CBCS

2021

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

 $[\mathbb{N}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$ denote the set of all natural, real, rational and complex numbers respectively.]

(Notations and symbols have their usual meanings).

- Answer *all* the following multiple choice questions. For each question 1 mark for choosing correct option and 1 mark for justification.
 - (a) Let (X, d) be a metric space and $A, B \subseteq X$. Choose the statement which is not true.
 - (i) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ (ii) $A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}$
 - (iii) $\partial(A \cap B) = \partial A \cap \partial B$ (iv) $d(A, B) = d(\overline{A}, \overline{B})$.

 $[\partial A \text{ denotes boundary of } A, d (A, B) \text{ denotes the distance between } A, B.]$

(b) Two metrices d and d^* are defined on \mathbb{R}^2 as follows :

For all
$$\hat{x} = (x_1, y_1), \hat{y} = (x_2, y_2) \in \mathbb{R}^2$$

$$d(\hat{x}, \hat{y}) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{1/2}$$

$$d^* (\hat{x}, \hat{y}) = \max\{ |x_1 - x_2|, |y_1 - y_2| \}.$$

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Then which of the following is true?

- (i) $d = d^*$ (ii) d and d^* are equivalent
- (iii) d and d* are not equivalent (iv) (\mathbb{R}^2, d^*) is a submetric space of (\mathbb{R}^2, d) .
- (c) Let $A = \{(x, y) : x, y \in \mathbb{R}, x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}\}$. Then which of the following is true with usual metric on \mathbb{R}^2 ?
 - (i) A is open but not compact in \mathbb{R}^2 (ii) A is not open but compact in \mathbb{R}^2
 - (iii) A is neither open nor compact in \mathbb{R}^2 (iv) A is both open and compact in \mathbb{R}^2 .

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(d) Consider \mathbb{R}^2 with usual metric.

Let
$$A = \left\{ \left(\frac{1}{n}, 0\right) \in \mathbb{R}^2 : n \in \mathbb{N} \right\} \cup \left\{ (0, 0) \right\}$$
 and
 $B = \left\{ \left(x, \sin \frac{1}{x}\right) \in \mathbb{R}^2 : 0 < x < 1 \right\}$. Then which of the following is correct?

(i) Both A, B are connected (ii) A is connected, B is disconnected

- (iii) B is connected, A is disconnected (iv) Both A, B are disconnected.
- (e) Let A be a subset of a metric space (X, d). Then $\{x \in X : d(x, A) = 0\}$ is :
 - (i) equal to \overline{A} and is compact (ii) equal to \overline{A} but not necessarily compact (iii) equal to A(iv) equal to $\{0\}$.
- (f) Let $T(z) = \frac{az+b}{cz+d}$ be a bilinear transformation. Then ∞ is a fixed point of T if and only if
 - (i) a = 0 (ii) b = 0
 - (iii) c = 0 (iv) d = 0.
- (g) Let $f(z) = |z|^2$; $z \in \mathbb{C}$. Then f is
 - (i) continuous everywhere but differentiable nowhere
 - (ii) differentiable only at z = 0
 - (iii) continuous nowhere
 - (iv) differentiable everywhere.

(h) The radius of convergence of the power series $\sum (4+3i)^n z^n$ is

- (i) 5 (ii) $\frac{1}{5}$
- (iii) 4 (iv) $\frac{1}{4}$.
- (i) Value of the integral $\int_C \sec(z) dz$, where C is the unit circle with centre at origin, is:
 - (i) 2 (ii) 0
 - (iii) 1 (iv) -5.

(2)

(j) Let $I = \int_{\gamma} z^2 dz$, where γ is along the real-axis from 0 to 1 and then along the line parallel to the

imaginary-axis from 1 to 1 + 2i. Which of the followings is true?

(i)
$$I = -\frac{11+2i}{3}$$

(ii) $I = \frac{11-2i}{3}$
(iii) $I = \frac{11+2i}{3}$
(iv) $I = \frac{11+2i}{3}$.

Unit - 1

(Metric Space)

Answer any five questions.

2. Let \mathbb{R}_{∞} be the extended set of real numbers. The function *d* defined by

 $d(x, y) = |f(x) - f(y)|, \forall x, y \in \mathbb{R}_{\infty}$, where f(x) is given by

$$f(x) = \begin{cases} \frac{x}{1+|x|}, & \text{when } -\infty < x < \infty \\ 1, & \text{when } x = \infty \\ -1, & \text{when } x = -\infty \end{cases}$$

Show that (\mathbb{R}_{∞}, d) is a bounded metric space.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences in a metric space (X, d). Write

$$x_n = d(a_n, b_n) \forall n \in \mathbb{N}$$

If $\{a_n\}$ is a Cauchy sequence and $x_n \to 0$ with respect to usual metric on \mathbb{R} , then prove that $\{b_n\}$ is a Cauchy sequence. Is this true if $\{x_n\}$ converges to nonzero limit? Justify. 3+2

4. Let (X, d) be a complete metric space and $\{F_n\}$ be a decreasing sequence of nonempty closed subsets

of X such that diam $(F_n) \to 0$ as $n \to \infty$. Then show that the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one

point. If diam $(F_n) \neq 0$ as $n \to \infty$, what would happen? Justify your answer. 3+2

Prove or disprove : Let (X, d) be a metric space and A be a closed and bounded subset of X. Then A is compact.

Please Turn Over

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- 6. (a) Prove that a metric space (X, d) having the property that every continuous map $f: X \to X$ has a fixed point, is connected.
 - (b) Let (X, d) be a complete metric space and $T: X \to X$ be a contraction on X. Then for $x \in X$, show that the sequence $\{T^n x\}$ is a convergent sequence. 2+3
- (a) Prove that the space Q of rational numbers with subspace metric of the usual metric of R is not connected.
 - (b) Prove that (a, b] is connected with usual metric of \mathbb{R} .
- 8. Let (X, d_1) and (Y, d_2) be two metric spaces and $f: (X, d_1) \rightarrow (Y, d_2)$ be uniformly continuous. Show that if $\{x_n\}$ is a Cauchy sequence in (X, d_1) then so is $\{f(x_n)\}$ in (Y, d_2) . Is it true if f is only continuous? Justify. 3+2
- 9. Let (X, d) be a complete metric space and let $f: X \to X$ be a contraction mapping with Lipschitz constant $t \ (0 \le t \le 1)$. If $x_0 \in X$ is unique fixed point of f, show that

$$d(x, x_0) \le \frac{1}{1-t} d\left(x, f(x)\right), \, \forall x \in X$$
5

2+3

Unit - 2

(Complex Analysis)

Answer *any four* questions.

10. (a) Let z_1 and z_2 be the images in the complex plane of two diametrically opposite points on the Riemann sphere under stereographic projection. Then show that

$$z_1 z_2 = -1$$
.

- (b) Prove or disprove : The image of the circle | z | = r (r ≠ 1) under the transformation w = z + 1/z is an ellipse.
- 11. Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \frac{(\overline{z})^2}{z}$ for $z \neq 0$ and f(0) = 0. Show that the Cauchy Riemann equations are satisfied at z = 0, but the derivative of f fails to exist there. 5
- 12. (a) If f is analytic function of $z = x + iy \in \mathbb{C}$ and $\overline{z} = x iy$, then show that $\frac{\partial f}{\partial \overline{z}} = 0$.
 - (b) Let f be analytic on a region G. If f assumes only real values on G, then show that f is a constant function. 3+2

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13. Let f be an analytic function in a region G. Show that Re(f) satisfies the following equation :

(5)

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0.$$
 5

14. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R > 0. Show that f is differentiable

on
$$|Z| < R$$
. Show that $\frac{df}{dz} = \sum_{n=1}^{\infty} na_n z^{n-1}$ and it has radius of convergence R . 5

15. (a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \frac{e^{in\pi}}{n}$.

(b) Let $f(z) = \overline{z}$ and γ is the semicircle from 1 to -1 passing through *i*. Evaluate $\int_{\gamma} f(z) dz$. 2+3

16. (a) Evaluate :
$$\int_{|z|=2} \frac{e^z + z^2}{z - 1} dz$$

(b) Show that $\left| \int_{\gamma} \frac{dz}{z^2 + 4} \right| \le \frac{\pi R}{(R^2 - 4)}$, where $\gamma(t) = \operatorname{Re}^{it}$ for $0 \le t \le \pi$ and $R > 2$. $3+2$