Subject : Mathematics

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Topic : CC9 , UNIT - II (Multivariate Calculus-II)

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Line integral

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Defn: Let $J = [a,b]$ be a finite closed interval in $\mathbb{R}^1$. A function $\alpha : J \to \mathbb{R}^n$ which is continuous ~~on~~ on J is called a continuous path in n-space. The path is called smooth if the derivative α' exists and is continuous on the open interval (a,b). The path is piecewise smooth if the interval $[a,b]$ can be partitioned into a finite no. of subinterval in each of which the path is smooth.

Derivative of vector fields

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ be a vector field. If a is an interior point of S. and if y is any vector in $\mathbb{R}^n$, we define $f'(a;y)$ by

$$f'(a;y) = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h}.$$

Whenever the limit exists, the derivative. $f'(a;y)$ is a vector in $\mathbb{R}^m$.

Let f_k be the k-th component of f. ie.

$$f = (f_1, f_2, \cdots f_k, f_{k+1}, \cdots f_m)$$

$\uparrow$ kth component.

the derivative $f'(a;y)$ exists iff $f_k'(a;y)$ exists for each $k = 1$ to m. in which we have.

$$f'(a;y) = (f_1'(a;y), \cdots, f_m'(a;y))$$

$$= \sum_{k=1}^{m} f_k'(a;y) \cdot e_k.$$

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Where e_k is the kth coordinate vector.

f is differentiable at an interior point a if there is a linear transformation $T_a : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a,v) \longrightarrow (1)$$

where $E(a,v) \to 0$ as $v \to 0$.

The first order Taylor formula holds for (1) with $\|v\| < r$ for some $r > 0$.

The term $E(a,v) \in \mathbb{R}^m$.

The linear transformation T_a is the total derivative of f at a.

Note: For scalar fields: $T_a(y)$ is the dot product of the gradient vector $\nabla f(a)$ and y.

For vector fields: $T_a(y)$ is a vector whose kth component is the dot product $\nabla f_k(a) \cdot y$.

Definition of Line Integral:

Let α be a piecewise smooth path in n-space defined on an interval $[a,b]$ and let f be a vector field defined and bdd on the graph of α.

The line integral of f along α is ~~defined~~ denoted by $\int f \cdot d\alpha$ and is defined by

$$\int f \cdot d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t)\, dt.$$

whenever the integral on RHS exists.

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If C denotes the graph of α, then $\int f\cdot d\alpha$ is written as $\int_C f\cdot d\alpha$. If $a = \alpha(a)$, $b = \alpha(b)$, then the line integral is written as $\int_a^b f$ or $\int_a^b f\cdot d\alpha$.

Note: The integral depends not only on a and b, but also on the path joining them.

When $a=b$, the path is said to be closed. The symbol $\oint$ is often used to indicate the integration along a closed path.

Explicitly

$$f = (f_1, \cdots, f_n) \quad \alpha = (\alpha_1, \cdots, \alpha_n).$$

then

$$\int f\cdot d\alpha = \int_a^b f(\alpha(t))\cdot \alpha'(t)\,dt$$

$$= \sum_{k=1}^{n} \int_a^b f_k(\alpha(t))\cdot \alpha_k'(t)\cdot dt.$$

In this case it is also written as $\int f_1 d\alpha_1 + \cdots + f_n d\alpha_n$.

In two dimension, the path α is given by the parametric equation,

$$x = \alpha_1(t), \; y = \alpha_2(t).$$

the line integral

$$\int_C f\cdot d\alpha = \int_C f_1 dx + f_2 dy$$

$$= \int_C f_1(x,y)dx + f_2(x,y)dy.$$

Problems:

1. Let f be a two dimensional vector field given by

$$f(x,y) = \sqrt{y}\,\hat{i} + (x^3+y)\,\hat{j} \;;\; \forall (x,y) \text{ with } y>0.$$

Calculate the line integral of f from $(0,0)$ to $(1,1)$ along each of the paths.

i) line with parametric eq. $x=t, y=t$, $0 \le t \le 1$.

ii) path with parametric eq. $x=t^2, y=t^3$, $0 \le t \le 1$.

2. i) $f(x,y) = (x^2-2xy)\hat{i} + (y^2-2xy)\hat{j}$ from $(-1,1)$ to $(1,1)$ along the parabola $y=x^2$.

ii) $f(x,y) = (2a-y)\hat{i} + x\hat{j}$ along the path $\alpha(t) = a(t-\sin t)\hat{i} + a(1-\cos t)\hat{j}$, $0 \le t \le 2\pi$.

iii) $f(x,y) = (y^2-z^2)\hat{i} + 2yz\hat{j} - x^2\hat{k}$ along $\alpha(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \le t \le 1$.

Basic properties of line integral:

1. Linearity: ~~$\int(af+bg) = a\int f\cdot d\alpha$~~

$$\int(af+bg)\cdot d\alpha = a\int f\cdot d\alpha + b\int g\cdot d\alpha.$$

2. Addivity:

$$\int_C f\cdot d\alpha = \int_{C_1} f\cdot d\alpha + \int_{C_2} f\cdot d\alpha.$$

where $C = C_1 \cup C_2$.

[... TO BE CONTINUED]

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