Study Material

Subject: Mathematics

Semester: 4th

Name of Teacher: Prabir Rudra

Topic: Mechanics (Central force field (contd.), Inverse square force field, Kepler's laws) (CC-10)

<u>Advice from faculty</u>

This is a continuation of the material that was shared on 11.04.2020. These notes carry both theory and problems on **Dynamics of a particle under a force field** particularly **a central force field (contd.)**. In addition to this it covers **Inverse square force field and Kepler's laws of motion**. The students are advised to read the relevant chapter from any standard text book they have at their disposal. In case they do not have any text book they can inform via the whatsapp group and I will try to provide an e-book. After reading the chapter from a text book, then concentrate on these notes to get a better understanding of the topics. In case you have any query regarding the topic you may consult me via e-mail (prudra.math@gmail.com) or whatsapp. **We are soon going to arrange an online doubt clearing session via video conference on the topic. The date and time will be informed soon.**

I think this will be sufficient material for 15 days at least. I will be back with the next topic after 10th May, 2020.

Date: 27/04/2020

When $r = a$, $\theta = 0$

$\therefore -\sin^{-1} 0 = c$

$\Rightarrow c = \pi$

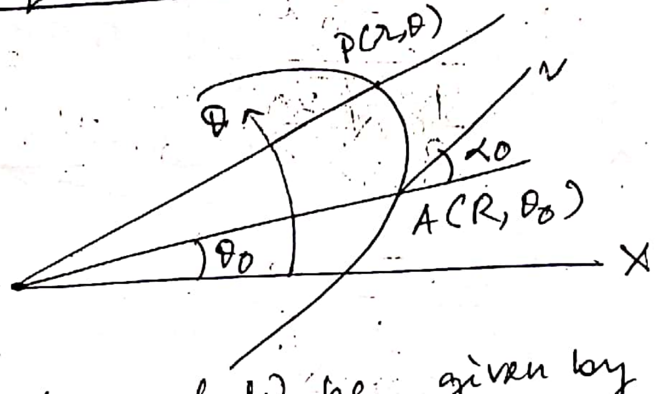
$$-\frac{r-a}{2a} = \sin(\pi+\theta)$$

$$\frac{r-a}{2a} = \sin\theta$$

or, $r = a(1+2\sin\theta)$

(iii) Inverse square force field:

$F = \frac{\mu}{r^2}$



Let the force field be given by $F = \frac{\mu}{r^2}$ directed to a fixed pt. O in the plane of motion. The differential equation of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} = \frac{\mu}{h^2}$

$$u = A\cos(\theta-\alpha) + \frac{\mu}{h^2} \quad \text{——— (2)}$$

When A and α are arbitrary constants to be determined from the initial conditions

or, $$\frac{1}{r}=\frac{\mu}{h^2}\left[1+\frac{Ah^2}{\mu}\cos(\theta-\alpha)\right]$$

or, $$\frac{h^2/\mu}{r}=1+\frac{Ah^2}{\mu}\cos(\theta-\alpha)$$

or, $$\frac{l}{r}=1+e\cos(\theta-\alpha) \quad \text{——— (3)}$$

Where $l=\frac{h^2}{\mu}$, $e=\frac{Ah^2}{\mu}$ ——— (4)

This equation is the polar equation of a conic referred to a focus of pole and whose axis is inclined at an angle α with the polar axis. The semi-latus rectum l & the ~~and~~ eccentricity of the conic are given by (4).

Thus the path described by the particle moving in an inverse square force field is a conic section having a focus at the centre of force.

Now, three different cases may arise:

From (2)

$$\frac{du}{d\theta} = -A\sin(\theta-\alpha)$$

$$\Rightarrow \left(u-\frac{\mu}{h^2}\right)^2\left(\frac{du}{d\theta}\right)^2 = A^2$$

$$\text{or, } u^2+\left(\frac{du}{d\theta}\right)^2 - \frac{2u\mu}{h^2} + \frac{\mu^2}{h^4} = A^2$$

$$\Rightarrow \frac{h^4}{\mu^2}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] - \frac{2h^2}{\mu}\cdot u = \frac{A^2h^4}{\mu^2} - 1$$

$$\text{or, } \frac{h^2v^2}{\mu^2} - \frac{2h^2}{\mu r} = \left(\frac{Ah^2}{\mu}\right)^2 - 1$$

$$\text{or, } \frac{h^2}{\mu^2}\left(v^2-\frac{2\mu}{r}\right) = e^2-1 \quad \text{— (5)}$$

If the particle be projected with a velocity V at a distance R from the pole we have from (5)

$$\frac{h^2}{\mu^2}\left(V^2-\frac{2\mu}{R}\right) = e^2-1$$

This shows that $e \lesseqgtr 1$

according as $V^2 \lesseqgtr \frac{2\mu}{R}$

Hence the orbit described by a particle is an ellipse, parabola or ~~a~~ hyperbola according as

$$v^2 \lesseqgtr \frac{2\mu}{R}$$

The equation ④ shows that eccentricity can be determined from the initial conditions.

<u>Eccentricity</u>, If the particle be projected from a pt. $A(R, \theta_0)$ with velocity V in a direction making an angle α_0 with OA then we have,

$$\frac{l}{R} = 1 + e\cos(\theta_0 - \alpha) \quad — ⑥$$

$$\frac{du}{d\theta} = \frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\dot{r}}{h}$$

$$\left(\frac{du}{d\theta}\right)_{at\,A} = -\frac{V\cos\alpha_0}{h}$$

$$l\frac{du}{d\theta} = -e\sin(\theta - \alpha) \text{ [From ③]}$$

$$\frac{l}{h}V\cos\alpha_0 = e\sin(\theta_0 - \alpha) \quad — ⑦$$

Again in a central orbit

$$vp = h$$

$$vR\sin\alpha_0 = h$$

$$l = \frac{h^2}{\mu} = \frac{v^2R^2\sin^2\alpha_0}{\mu}$$

$$(6)^2 + (7)^2 \quad \left(\frac{l}{R} - 1\right)^2 + \frac{l^2v^2\cos^2\alpha_0}{h^2} = e^2$$

or, $$e^2 = \frac{l^2}{R^2} - \frac{2l}{R} + \frac{l^2v^2\cos^2\alpha_0}{v^2R^2\sin^2\alpha_0} + 1$$

$$= \frac{l^2}{R^2} - \frac{2l}{R} + 1 + \frac{l^2}{R^2}\cot^2\alpha_0$$

$$= \frac{l^2}{R^2}\operatorname{cosec}^2\alpha_0 - \frac{2l}{R} + 1$$

$$= 1 + \frac{v^4R^2\sin^2\alpha_0}{\mu^2} - \frac{2v^2R\sin^2\alpha_0}{\mu}$$

$$= 1 + \frac{v^2R^2\sin^2\alpha_0}{\mu^2}\left(v^2 - \frac{2\mu}{R}\right) \qquad (8)$$

This relation determines e from the initial conditions and it shows that the central orbit is an ellipse parabola or hyperbola according as $v \gtreqless \frac{2\mu}{R}$

<u>Note</u>: The nature of the conic is independent of the angle of projection but it depends on the magnitude of velocity of projection.

PEDAL FORCE

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

$$\frac{h^2}{p^3} dp = \mu \frac{dr}{r^2}$$

$$\Rightarrow -\frac{h^2}{2p^2} = -\frac{\mu}{r} + c$$

i.e., $$\frac{h^2}{p^2} = \frac{2\mu}{r} + A \quad \text{—— (1)}$$

or, $$\frac{l}{p^2} = \frac{2}{r} + \frac{A}{\mu} \quad \text{where } l = \frac{h^2}{\mu} \quad \text{—— (2)}$$

Now we know that the pedal equation of the ellipse, parabola and hyperbola w.r.t. focus as pole are respectively

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \qquad p^2 = ar$$

And $$\frac{b^2}{p^2} = \frac{2a}{r} + 1 \quad \Bigg| \quad \frac{l}{p^2} = \frac{2}{r} \mp \frac{1}{a} \quad \text{——} (3)$$

$$\frac{l}{p^2} = \frac{2}{r} \quad \text{——} (4)$$

Comparring these equations with equation (2) we see that the path of the particle is a conic with centre of force as focus, and it is an ellipse or parabola or hyperbola according as $A \lesseqgtr 0$

Now $pv = \mu$
So from (1) we get

$$v^2 = \frac{2\mu}{r} + A$$

$$\cancel{or\ A = v^2 - \frac{2\mu}{r}}$$

If the particle is projected from a distance R from the centre of force with a velocity v in any direction we have $v^2 = \frac{2\mu}{R} + A$

or, $A = v^2 - \frac{2\mu}{R}$

$\therefore$ A is negative, zero or positive according as $v^2 \lesseqgtr \frac{2\mu}{R}$

Hence, the required orbit is an ellipse, parabola or hyperbola according as $v^2 \lesseqgtr \frac{2\mu}{R}$

<u>Note</u>:– As we know that the velocity from infinity under the inverse a square law is $\sqrt{\frac{2\mu}{R}}$ at a distance R from the centre of force the above result many be stated in the form, the path of a particle moving in a plane under law of inverse square to a fixed is an ellipse, parabola or hyperbola. according as the velocity at any point is less than equal or greater than velocity from infinity to that point

~~Vel~~

Comparison of eqn. 2 with 3 & 4

$A = -\frac{\mu}{a}, 0, \frac{\mu}{a}$ according as the path is ellipse, parabola or hyperbola.

$$\therefore v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Ellipse

$$= \frac{2\mu}{r} + \frac{\mu}{a} = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

hyperbola

$$= \frac{2\mu}{r} \rightarrow \text{parabola}$$

The converse of the result is also true. It can be stated that when a particle describes a conic under a force which is always directed to a focus, the law of force is that of inverse square.

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Kepler's laws of planetary motion:

(i) Each planet describes an ellipse with the sun at one of its foci.

(ii) The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time.

(iii) The squares of the periodic times of the planets are in the ratio of the cubes of the semi-major axes of their elliptical orbits.

Newton's laws of gravitation from kepler's laws:

From kepler's 1st law the orbit of a planet is an ellipse having one of its foci at the sun. Referred to focus as pole and ~~the~~ polar axis along the major axis of the ellipse the equation of the ellipse can be taken as

$$\frac{L}{r} = 1 + e\cos\theta.$$

If (r,θ) be any point on the ellipse the the rate at which the area is described about the focus is $\frac{1}{2}r^2\dot\theta$ which is constant by Kepler's second Law.

$$\therefore \frac{1}{r}\frac{d}{dt}\left(\frac{1}{2}r^2\dot\theta\right)=0$$

$$f_\theta=0$$

Which shows that the force acting on the planet is towards the pole. If $F=f_r$ be the central force per unit mass of the planet, then the diff. equn. of the orbit is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$$

Now $u=\frac{1}{r}$, $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$,

$$=-\frac{e}{L}\sin\theta$$

$$\frac{d^2u}{d\theta^2}=\frac{e}{L}\cos\theta$$

$$\therefore \frac{F}{h^2u^2}=-\frac{e}{L}\cos\theta+\frac{1}{L}+\frac{e}{L}\cos\theta=\frac{1}{L}.$$

$\Rightarrow F = \frac{h^2 u^2}{l} = \frac{\mu}{r^2}$ where $\mu = \frac{h^2}{l}$

The ~~forces~~ force on the planet depends on r alone and varies inversely as the square of the distance from the sun.

Since, the periodic time of the planet $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$

$$\therefore \frac{T^2}{a^3} = \left(\frac{2\pi}{\sqrt{\mu}}\right)^2$$

But by Kepler's law $\frac{T^2}{a^3}$ is same for all planets. Hence μ ~~goes~~ does not depend on the mass of the planet.

It depends on the mass of the sun only. Putting $\mu = GM$, the force between the sun and the planet is

$$\frac{\mu}{r^2} = \frac{GM}{r^2}$$ per unit mass of the planet.

$$\therefore F = \frac{GMm}{r^2}$$

Hence follows the universal Law of Gravitation. On the other hand these laws can be obtained as consequences of Newton's gravitational law.

In the case of planetary motion the force of attraction is that exerted by the sun on the planet.

If M be the mass of the sun then the force of attraction per unit mass of the planet is $\frac{GM}{r^2}$ where r is the distance between sun and the planet.

Hence the equation of the path is

$$\frac{h^2/GM}{r} = 1 + \frac{Ah^2}{GM}\cos(\theta - \alpha)$$

Which is an ellipse with the sun at one of $[\because \mu = G.M]$ the foci and ~~the~~ eccentricity is equal to $\frac{Ah^2}{GM}$ $\left(\text{if } A < \frac{GM}{h^2}\right)$ and semi-latus rectum is $\frac{h^2}{GM}$.

This gives Kepler's first law

Again the conservation of angular momentum in a central force field motion gives the second law. Moreover if T be the period of revolution of the planet round the sun then,

$$T = \frac{\text{area of the elliptic orbit}}{\text{areal velocity}}$$

$$= \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}$$

Where a & b are the lengths of the semi-axes of the ellipse.

Hence $\frac{b^2}{a} = \frac{h^2}{GM}$

i.e., $\frac{a^2b^2}{h^2} = \frac{a^3}{GM}$

$\therefore \quad T^2 = \frac{4\pi^2 a^2 b^2}{h^2} = \frac{4\pi^2 a^3}{GM}$

i.e. $T^2 \propto a^3$ which proves the third law. $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$.

Modification of Kepler's third laws

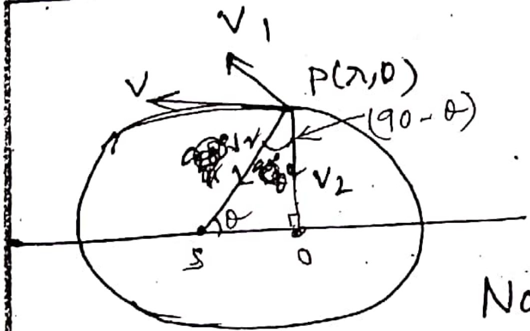
In the preceding discussion we have considered the acc^n. produced by the action of the sun on a planet and ignored the acc^n. of the sun produced by the action of the planet. Now the acc^n. of the sun to the planet is $\frac{Gm}{r^2}$ and that of the planet to the sun is $\frac{GM}{r^2}$. So the acc^n of the planet to the sun is of the planet $\frac{GM}{r^2}$. So the acc^n. of the planet relative to the sun is $\frac{G(M+m)}{r^2}$.

Hence the periodic time for the relative motion of the planet is given by $T = \frac{2\pi\, a^{3/2}}{\sqrt{G(M+m)}}$, where a is the semi-major axis of the relative orbit of the planet.

PROBLEM I :

If a particle describes an ellipse under a force towards a focus show that the velocity in the ellipse can be resolved into two components of constant magnitude are perpendicular to the major axis.

Solⁿ.



$$\dot{r} = -v_2 \sin\theta$$

$$r\dot{\theta} = v_1 - v_2 \cos\theta$$

$$r^2 \dot{\theta} = h$$

Now the equⁿ. of the ellipse i.e.,

$$\frac{l}{r} = 1 - e\cos\theta,$$ with focus as pole.

or, $$-\frac{l}{r^2}\cdot\dot{r} = e\sin\theta\cdot\dot{\theta}$$

$$\dot{r} = -\frac{eh}{l}\sin\theta$$

$$\Rightarrow v_2 = -\frac{\dot{r}}{\sin\theta} = \frac{eh}{l}$$

$$r\dot{\theta} = \frac{h}{r} = v_1 - \frac{eh}{l}\cos\theta$$

$$v_1 = \frac{h}{r} + \frac{eh}{l}\cos\theta$$

$$= h\left(\frac{1-e\cos\theta}{h}\right) + \frac{eh}{l}\cos\theta$$

$$= \frac{h}{l} \text{ constant.}$$

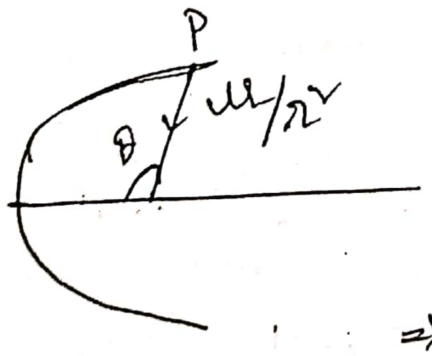
Hence for very position of the particle V_1 & V_2 are constant.

Problem 2:

Prove that the time required to describe an arc of a parabola under the action of the force $\frac{\mu}{r^2}$ to the focus starting from the nearer end of axis is given by

$$\sqrt{\frac{2a^3}{\mu}}\left[\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right]$$

where 'a' is the apsidal distance and 'θ' is measured from the axis.



For a certain orbit $r^2\dot{\theta} = h$

$$\Rightarrow r^2\frac{d\theta}{dt} = h$$

$$\Rightarrow h\,dt = r^2 d\theta$$

$$\Rightarrow h\int_0^t dt = \int_0^\theta r^2 d\theta$$

$$\frac{L}{r} = 1 + \cos\theta$$

$$\Rightarrow r = \frac{L}{1+\cos\theta}$$

$$\Rightarrow r^2 = \frac{l}{(1+\cos\theta)^2}$$

$$\Rightarrow ht = \int_0^\theta \frac{l^2}{(1+\cos\theta)^2} d\theta$$

$$\Rightarrow ht = \frac{1}{4}\int_0^\theta l^2 \sec^4\frac{\theta}{2} d\theta$$

$$= \frac{l^2}{4}\int_0^\theta (1+\tan^2\frac{\theta}{2}) \sec^2\frac{\theta}{2} d\theta$$

$$= \frac{l^2}{2}\left[\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right]$$

$$\Rightarrow t = \frac{l^2}{2h}\left[\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right]$$

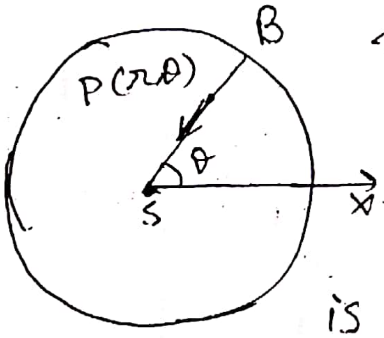
$$= \sqrt{\frac{2a^3}{\mu L}}\left[\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right]$$

$$\left[\frac{l^2}{2\sqrt{\mu L L}} = \frac{l^{3/2}}{2\sqrt{\mu L}} = \frac{2^{3/2} a^{3/2}}{2\sqrt{\mu L}} = \sqrt{\frac{2a^3}{\mu L}}\right]$$

Problem:

If a planet were suddenly stopped in its orbit supposed circular, show that it will fall ~~fall~~ into the ~~sum~~ sun in a time which is $\frac{\sqrt{2}}{8}$ time the period of the planets revolution

Soln Let a be the radius of the circular orbit. Let the planet be stopped at B such that $\angle BSX = \theta$. The diff equn. of the motion of the planet gives



$$\frac{d^2 r}{dt^2} = -\frac{\mu}{r^2}$$

where $P(r,\theta)$ is the position of the planet after time 't' elapsed from the position B.

$$\left(\frac{dr}{dt}\right)^2 = \frac{2\mu}{r} + c$$

At B, $\frac{dr}{dt} = 0$, $r = a$

$$c = -\frac{2\mu}{a}$$

$$\therefore \left(\frac{dr}{dt}\right)^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)$$

$$\text{or,}\quad \frac{dr}{dt} = -\frac{2\mu}{a}\frac{\sqrt{a-r}}{\sqrt{r}}$$

$$\Rightarrow -\int_a^0 \sqrt{\frac{r}{a-r}}\,dr = \int_0^{t_1}\sqrt{\frac{2\mu}{a}}\,dt$$

$$\Rightarrow \sqrt{\frac{2\mu}{a}}\,t_1 = \int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\,2a\sin\theta\cos\theta\,d\theta$$

$$= a\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= a\left[\pi/2\right] = \frac{\pi a}{2}$$

$$t_1 = \frac{\pi a^{3/2}}{2\sqrt{2\mu}}$$

If T is the time period of revolution $\frac{h}{2}T = \pi a^2$

$$T = \frac{2\pi a^2}{h} = \frac{2\pi a^2}{\sqrt{\mu a}} = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$$

$$\Rightarrow \frac{t_1}{T} = \frac{\pi}{2\sqrt{2\mu}} \times \frac{\sqrt{\mu}}{2\pi}$$

$$= \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$$

$$\therefore t_1 = \frac{\sqrt{2}}{8}T$$

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A particle is projected at right angle to the line joining it to a centre of force attracting according to the law of inverse square of the distance, with a velocity $\frac{\sqrt{3}}{2}V$ where V denotes the velocity from infinity. Find the eccentricity of the orbit described and show that the periodic time is $2\pi T$, T the periodic time taken to describe the major axis of the orbit with velocity V.

Soln: Let R be the distance of the point of projection from the centre of force.

Then as $v\frac{dv}{dr} = -\frac{\mu}{r^2}$ we have

$$\int_0^V v\,dv = -\int_\infty^R \frac{\mu}{r^2}\,dr$$

, V being the velocity from infinity at the distance R,

i.e. $V^2 = \frac{2\mu}{R}$

As the velocity of projection V' is given by $V' = \frac{\sqrt{3}}{2} V$

we have $V'^2 = \frac{3}{4} V^2 = \frac{3}{4} \frac{2\mu}{R} < \frac{2\mu}{R}$

So the particle will describe an ellipse having a focus at the centre of force.

$\therefore V^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right)$

Initially $V'^2 = \mu \left(\frac{2}{R} - \frac{1}{a}\right)$

or, $\frac{3}{4} \frac{2\mu}{R} = \mu \left(\frac{2}{R} - \frac{1}{a}\right)$

$\Rightarrow \frac{3}{2R} - \frac{2}{R} = -\frac{1}{a}$

$\Rightarrow \frac{1}{a} = \frac{4}{2R} - \frac{3}{2R} = \frac{1}{2R}$

or, $a = 2R$

As the initial direction of projection is perpendicular to the radius vector, from the relation

$Vp = h = \sqrt{\mu l}$ we get,

$V'^2 R^2 = \mu l = \mu \frac{b^2}{a} = \mu a \frac{b^2}{a^2}$

$= \cancel{} \mu a (1 - e^2)$

or, $\frac{3}{4} \frac{2\mu}{R} R^2 = \mu a (1 - e^2)$

or, $\frac{3}{2}R = 2R(1-e^2)$ $[\because a = 2R]$

or, $\frac{3}{4} = 1 - e^2$ or $e^2 = \frac{1}{4}$

$e = \frac{1}{2}$

As T is the time to describe the major axis of the orbit with the velocity V, we have

$$V.T = 2a$$

$$\Rightarrow T = \frac{2a}{V} = \frac{2a}{\sqrt{\frac{2\mu}{R}}}$$

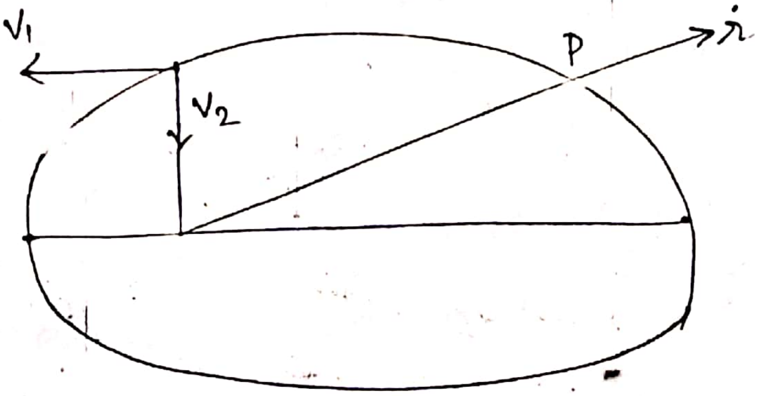
$$= \frac{2a}{\sqrt{\frac{4\mu}{a}}} = \frac{a^{3/2}}{\sqrt{\mu}}$$

Now time period of the particle $= \frac{2\pi}{\sqrt{\mu}} a^{3/2} = 2\pi T.$

PROBLEM :

A planet is describing an ellipse about the sun as focus. Show that the velocity ~~away~~ away from the sun is greatest when the radius vector ~~to~~ to the planet is at right angle to the major axes and that it is then $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where 2a is the major axis and e is eccentricity

$\frac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis and e is eccentricity



Ans: $\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$

$r\dot{\theta}^2 = \frac{\mu}{r^2}$ [When velocity is maximum]

or, $(r^2\dot{\theta})^2 = \mu r$

i.e., $h^2 = \mu r$

again $h^2 = \mu l$

$\therefore r = l$

$\therefore$ The velocity will be max. when the planet will be at one end of the latus rectum.

Velocity component $\perp$ to the major axis is constant and is equal to $\frac{eh}{\chi}$

$$= e\sqrt{\frac{\mu}{l}} = e\,\frac{2\pi}{T}\,a^{3/2}\,\frac{1}{\sqrt{a}\,(1-e^2)^{1/2}}$$

$$= \frac{2\pi a e}{T(1-e^2)^{1/2}}$$

(2) An artificial satellite can be projected vertically from the surface of the earth supposed spherical and of radius R. with a velocity just sufficient to carry it to a height 5R from the earth's centre. How far from the centre will the satellite reach if it is projected horizontally from the earth's surface with the same velocity.

<u>Ans:</u>

$$\int_V^0 v\,dv = -\int_R^{5R} \frac{\mu}{r^2}\,dr$$

$$\Rightarrow -\frac{v^2}{2} = \left[\frac{\mu}{r}\right]_R^{5R}$$

$$\Rightarrow 0 - \frac{v^2}{2} = \frac{\mu}{5R} - \frac{\mu}{R}$$

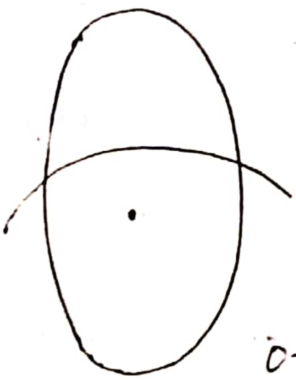
$$= -\frac{4\mu}{5R}$$

$$\Rightarrow v = \sqrt{\frac{8}{5}\frac{\mu}{R}}$$

$$\Rightarrow v^2 = \frac{8\mu}{5R} < \frac{2\mu}{R}$$

Now $\mu = gR^2$

$$\therefore v = \sqrt{\frac{8}{5} gR}$$



Hence the satellite describes an ellipse with a focus on the earth's centre if $2a$ be the length of the major.

Then velocity of the satellite at any time is

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

with $$v^2 = \mu\left[\frac{2}{R} - \frac{1}{a}\right]$$

$$\frac{8gR}{5} = gR^2\left[\frac{2}{R} - \frac{1}{a}\right]$$

$$= 2gR - \frac{gR^2}{a}$$

or, $\frac{R}{a} = 2 - \frac{8}{5} = \frac{2}{5}$

or, $a = \frac{5R}{2}$

As the initial projection is horizontal

$vR = h$

$\Rightarrow v^2R^2 = h^2 = \mu l = gR^2 a(1-e^2)$

or, $\frac{8gR}{5} = \frac{5gR}{2}(1-e^2)$

or, $e = \frac{3}{5}$

The maximum distance is attained at the further end of the major axis.

$D_{max} = a + ae$

$= \frac{5R}{2} + \frac{5R}{2} \cdot \frac{3}{5}$

$= \frac{5R}{2} + \frac{3R}{2} = 4R$

is

attained

ie

(3) A particle moving in a ellipse under the action of a force towards the focus at 'O' moves from the greatest distance from O to an extremity of the minor axis in time 't' and then to the least distance from O in time $\frac{t}{k}$. Prove that the eccentricity of the ellipse is $\frac{\pi}{2}\left(\frac{k-1}{k+1}\right)$

Soln

$$t = \left(\frac{\pi ab}{4} + \frac{1}{2}aeb\right) / \frac{h}{2} \quad \text{——— (1)}$$

$$t/k = \left(\pi ab/4 - \frac{1}{2}aeb\right) / \frac{h}{2} \quad \text{——— (2)}$$

(1) ÷ (2)

$$k = \frac{\pi + 2e}{\pi + 2e}$$

$$\frac{k-1}{k+1} = \frac{\pi + 2e - \pi + 2e}{\pi + 2e + \pi - 2e}$$

$$e = \frac{\pi}{2}\left(\frac{k-1}{k+1}\right)$$