Application of vector analysis to alternating Current ckts

Introduction:- Representation of instantaneous value of alternating current as a scalar product.

Let $\vec{I}$ be a vector of constant magnitude $\hat{I}$, rotating in positive (counter-clockwise) sense with constant angular velocity 'ω', its slope, relative to unit vector $\vec{v}$, at time $t=0$ being 'ϕ'.

'i' be instantaneous current $\hat{I}\cos(\omega t+\phi)$.

Since at time 't' the vector $\vec{I}$ makes an angle $(\omega t+\phi)$ with $\vec{v}$.

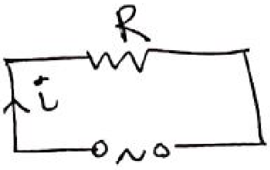
$\therefore\ i = \vec{I}\cdot\vec{v} = \hat{I}\exp(j(\omega t+\phi))\ \vec{v}\cdot\vec{v}$.

$\therefore$ An alternating current of amplitude $\hat{I}$ and angular freq ω can be represented as a scalar product with $\vec{v}$ of a vector $\vec{I}$, ~~makes an angle $(\omega t+\phi)$ with $\vec{v}$~~. of constant magnitude $\hat{I}$, rotating with angular velocity 'ω', initial position of $\vec{I}$ making an angle ϕ with $\vec{v}$.

Current and potential relationship in vector form

a) Pure resistance (R)

Suppose alternating potential difference $\hat{V}\cos(\omega t)$ applied to non-inductive resistance R

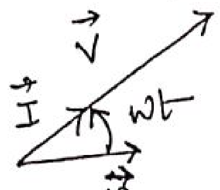


Then 'i' $\rightarrow$ instantaneous value of current

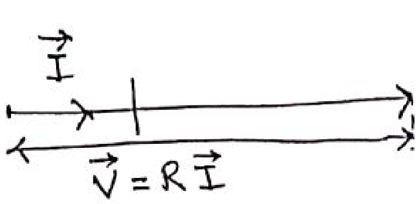
$Ri = V$

or $R\,\vec{I}\cdot\vec{v} = \vec{V}\cdot\vec{v}$ | taking projection along $\hat{v}$.

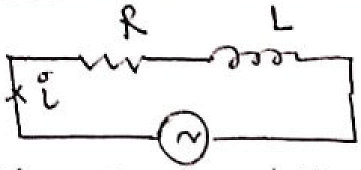
$\therefore\ \vec{V} = R\vec{I}$.



Voltage and current are in phase



Inductive Resistance (R and L in series) :—



Now, $v = iR + L\left(\frac{dI}{dt}\right)$

$\vec{V}\cdot\vec{v} = \hat{v}\, \exp(j\omega t)\ \vec{v}\cdot\vec{v}$

$= R\,\vec{I}\cdot\vec{v} + j\omega L\ \vec{I}\cdot\vec{v}$

$\therefore \vec{I} = \dfrac{\hat{v}\,\exp(j\omega t)\ \vec{v}}{(R+j\omega L)}$

$= \dfrac{\vec{V}}{Z}.$

Where, $Z = R + j\omega L$ is known as vector impedance or impedance operator.

Let, $v = \hat{v}\cos(\omega t)$

$= \hat{v}\ \exp(j\omega t)\ \vec{v}\cdot\vec{v}$

$= \vec{V}\cdot\vec{v}$

$i = \vec{I}\cdot\vec{v}$

Let, $\dfrac{1}{(R+j\omega L)}$

$= \dfrac{R - j\omega L}{(R^2+\omega^2L^2)} = \alpha - j\beta = \rho\ \exp(-j\varphi)$

$\alpha = \dfrac{R}{(R^2+\omega^2L^2)}$, $\beta = \dfrac{\omega L}{(R^2+\omega^2L^2)}$. $\tan\varphi = \left(\dfrac{\omega L}{R}\right)$

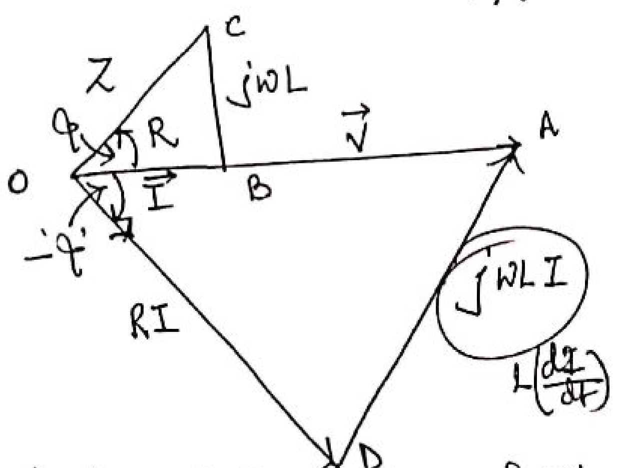
$\therefore \vec{I} = \dfrac{\hat{v}\ \exp(j\omega t)}{(R+j\omega L)}\vec{v}.$

$= \dfrac{\hat{v}}{\sqrt{R^2+\omega^2L^2}}\exp(j\omega t)\,.\exp[-j\varphi]\ \vec{v}$

$= \dfrac{\hat{v}}{\sqrt{R^2+\omega^2L^2}}.\exp\left[j(\omega t - \varphi)\right]\ \vec{v}.$

$\therefore\ i = \vec{I}\cdot\vec{v} = \vec{I}_{\omega,\varphi}\,\vec{v}$

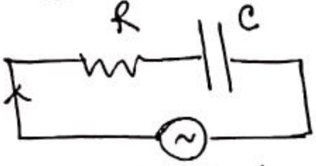
Here $\vec{I}$ is a vector of magnitude $\hat{I} = \dfrac{\hat{v}}{\sqrt{R^2+\omega^2L^2}}$ rotating with angular velocity ω, and whose angular position at $t=0$ is, $-\tan^{-1}\left(\dfrac{\omega L}{R}\right)$ i.e. phase of current is always behind impressed voltage by amount 'φ'.



'ωL' is inductive reactance of coil. $\vec{OB}$ horizontally equal to 'R'. 'BC' normal to AB → length ωL is '$j\omega L$'. $\Rightarrow$ $\vec{OD} + \vec{DA} = \vec{OA} = \vec{V}$, $\vec{OD} = \vec{I}R$, $\vec{DA} = j\omega L\vec{I}$

$\vec{OD}$ and $\vec{DA}$ are voltage vectors gives potential difference betn R and L

Imperfect condenser i.e. R and C in series



$\vartheta = \hat{v}\cos(\omega t)$

Imperfect condenser may be regarded as Capacitance 'C' in series with resistance 'R'

$$\vartheta = Ri + \frac{q}{C}$$
$$= Ri + \frac{1}{C}\int i\,dt$$

or $\vec{V}\cdot\vec{\vartheta} = R\vec{I}\cdot\vec{\vartheta} + \frac{1}{j\omega C}\vec{I}\cdot\vec{\vartheta}$

as $i = \hat{I}\exp j(\omega t)\vec{\vartheta}$

$\int i\,dt = \frac{\hat{I}}{j\omega}\exp j(\omega t)\vec{\vartheta}$

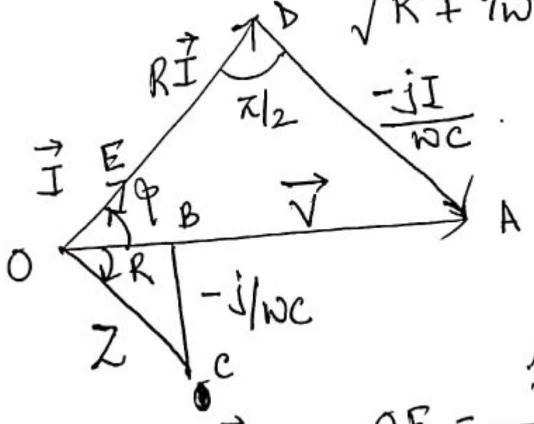
$$\therefore \vec{V} = R\vec{I} + \frac{1}{j\omega C}\vec{I}$$
$$= \left(R - \frac{j}{\omega C}\right)\vec{I}$$

$$\therefore \vec{I} = \frac{\vec{V}}{(R - j/\omega C)} = \frac{(R + j/\omega C)}{(R^2 + 1/\omega^2 C^2)}\vec{V}$$

$$= \frac{\hat{V}}{\sqrt{R^2 + 1/\omega^2 C^2}}\cdot\left[\exp j(\omega t + \varphi)\right]\vec{\vartheta}$$

where $\vec{V} = \hat{V}e^{(j\omega t)}$

$$\therefore i = \vec{I}\cdot\vec{\vartheta} = \frac{\hat{V}}{\sqrt{R^2 + 1/\omega^2 C^2}}\cdot\cos(\omega t + \varphi)$$ and $\tan\varphi = \left(\frac{1}{\omega CR}\right)$



Horizontal to represent $\vec{V}$.

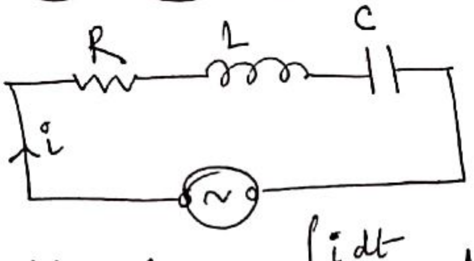
$OB = R$, $BC = -j/\omega C$ (vertically downward)

$OC = Z = (R - j/\omega C)$

Also, $\vec{I} = \frac{\vec{V}}{Z}$, $OE = \frac{\hat{V}}{\sqrt{R^2 + 1/\omega^2 C^2}}$ and it is advance of OA by φ.

$OD = R\vec{I}$, $DA = -j/\omega C\ I$

Resistance, inductance and Capacitance in Series :—



$$v = iR + \frac{\int i\,dt}{C} + L\left(\frac{di}{dt}\right)$$

$$\therefore v = \vec{V}\cdot\vec{v} = \hat{v}\left(\exp(j\omega t)\right)\vec{v}\cdot\vec{v}.$$

$$i = \vec{I}\cdot\vec{v}\ ,\quad \vec{I} \text{ has to be determined.}$$

$$\hat{v}\left[\exp(j\omega t)\right]\vec{v}\cdot\vec{v} = \left(R + j\omega L + \frac{1}{j\omega C}\right)\vec{I}\cdot\vec{v}$$

$$\therefore \vec{I} = \frac{\hat{v}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}\left(e^{j\omega t}\right)\vec{v}$$

$$= \frac{\left\{R - j\left(\omega L - 1/\omega C\right)\right\}\hat{v}}{\left[R^2 + \left(\omega L - 1/\omega C\right)^2\right]}\exp(j\omega t)\,\vec{v}$$

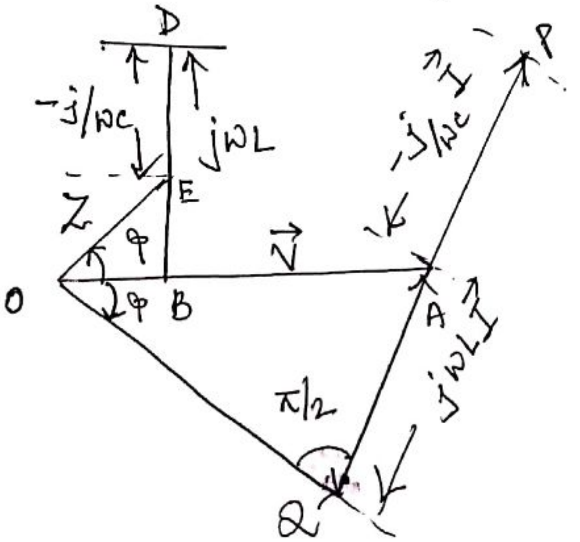
$$\therefore \vec{I} = \frac{\hat{v}}{\sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}}\cdot\left[\exp j(\omega t - \varphi)\right]\vec{v}$$

$$\varphi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right),$$

Thus $\vec{I}$ is a vector of magnitude

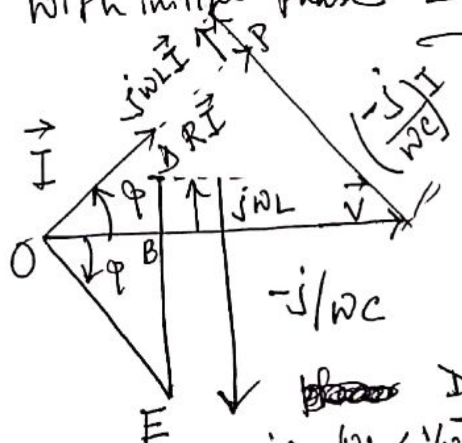
$$\hat{I} = \frac{\hat{v}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

rotating with angular velocity 'ω' with initial phase '$-\varphi$'



If $(\omega L) > (1/\omega C)$

phase of Current is always behind impressed voltage

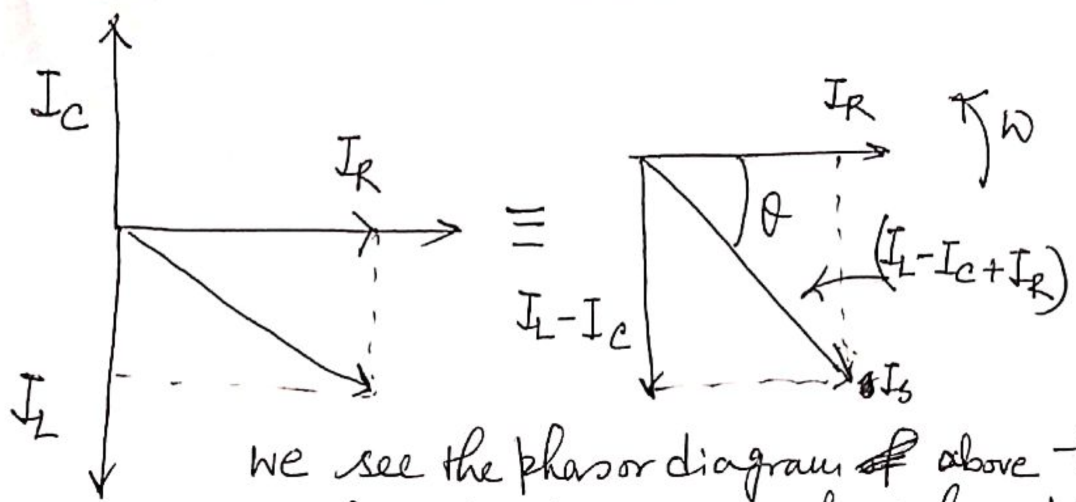


~~phase~~ $DE > BD$

ie $(\omega L < 1/\omega C)$

ie φ is $-$ve, Current leads voltage.

Phasor Diagram of Parallel LCR Ckt



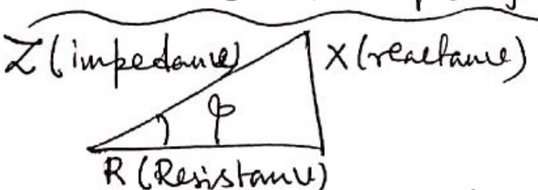
We see the phasor diagram above that the current vectors produce a rectangular triangle, comprising of hypotenuse 'I_S', horizontal axis 'I_R' and vertical axis $(I_L - I_C)$. It forms a Current triangle
We can use pythagoras's theorem on this current triangle to mathematically obtain the magnitude of branch currents along x-axis and y-axis.
then determine total current 'I_S' of these components as shown.

Current Triangle for Parallel RLC ckt :-

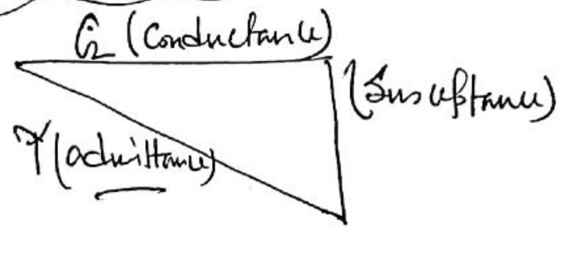
$$I_S^2 = I_R^2 + (I_L - I_C)^2$$

$$\therefore I_S = \sqrt{\left(\frac{V}{R}\right)^2 + \left(\frac{V}{X_L} - \frac{V}{X_C}\right)^2} = \frac{V}{Z}$$

Admittance triangle for parallel RLC ckt :-

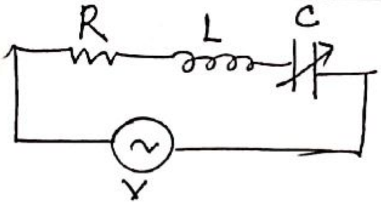


Impedance triangle



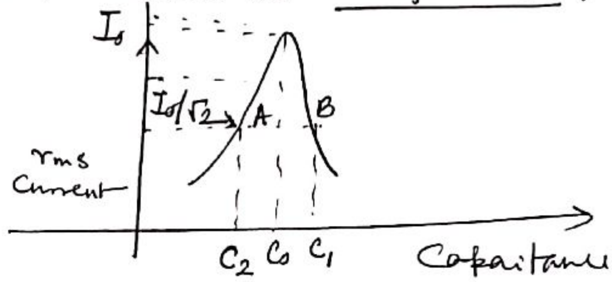
$$Z = \frac{1}{Y} = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L} - \omega C\right)^2}}$$

Series - Resonance (Current Resonance) :—

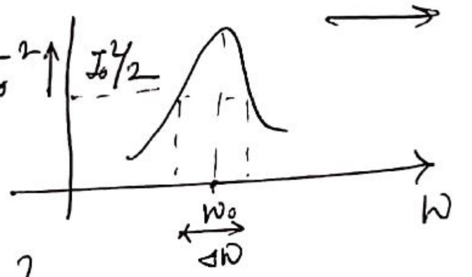


'v' be rms potential difference applied to simple Series resonance ckt
ckt is Known as acceptor ckt, (with variable C)

$$I^2 = \frac{V^2}{R^2 + (WL - 1/WC)^2}$$



$$\therefore I^2 = \frac{V^2}{R^2} \times \frac{1}{1 + \frac{1}{R^2}\left[\frac{1}{W_0 C_0} - \frac{1}{W_0 C}\right]^2}$$



at resonance

$$W_0 L = \frac{1}{W_0 C_0} \Rightarrow W_0 = \frac{1}{\sqrt{L C_0}}$$

$I_0 = V/R \rightarrow$ Current at resonance

$$\therefore \left(\frac{I_0}{I}\right)^2 - 1 = \frac{1}{W_0^2 R^2}\left[\frac{1}{C_0} - \frac{1}{C}\right]^2$$

choosing, $(I_0/I)^2 = 2$

$$\therefore W_0^2 R^2 = \left(\frac{1}{C_0} - \frac{1}{C}\right)^2 = \left(\frac{C - C_0}{C C_0}\right)^2.$$

$$\therefore W_0 R = \frac{(C_1 - C_0)}{C_0 C_1} \text{ or } \frac{(C_0 - C_2)}{C_0 C_2}$$

where, C_1 and C_2 are values of C

and, $1/C_1 + 1/C_2 = 2/C_0$

In practice, C_1 and C_2 dont differ very much. from C_0

$$\therefore C_0 C_1 = C_0 C_2 = C_0^2$$

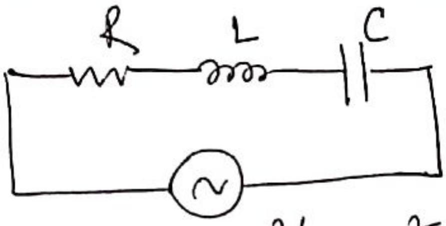
$$\therefore \frac{C_1 - C_2}{C_0^2} = 2 W_0 R$$

$$\frac{(C_1 - C_2)}{2 C_0} = W_0 C_0 R = \frac{R}{W_0 L}$$

$\frac{2C_0}{C_1 - C_2}$ is measure of peaking of response curve ie selectivity of ckt. denoted by 'Q'

$$\therefore \boxed{Q = \text{Quality factor / Magnification} = \frac{W_0 L}{R}}$$ ✓

Series Resonance (Current resonance)



$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V/L}{\sqrt{\left(\omega - \frac{1}{\omega LC}\right)^2 + R^2/L^2}}$$

Now, $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\therefore I = \frac{V/L}{\sqrt{\frac{(\omega^2 - \omega_0^2)^2}{\omega^2} + R^2/L^2}}$$

or, $$I^2 = \frac{V^2/L^2 \; \omega^2}{\left[(\omega_0^2 - \omega^2)^2 + \alpha^2 \omega^2\right]}$$

$\therefore R^2/L^2 = \alpha^2$,

If 'I_0' be current at resonance,

$$\therefore I_0 = \sqrt{\frac{V^2/L^2 \cdot \omega_0^2}{R^2/L^2 \; \omega_0^2}} = \frac{V}{R}.$$

In order to have, $I_0^2/I^2 = 2$, let the frequency of the supply be such that, $\omega = \omega_0 \pm \frac{1}{2}\Delta\omega$

$(\omega - \omega_0) \times 2 = \Delta\omega$

$$\cancel{\therefore I^2 = \frac{V^2/L^2 \cdot \omega_0^2}{\left[\alpha^2(\omega_0 \pm \frac{1}{2}\Delta\omega)^2 + (\omega_0 + \omega)^2(\omega_0 - \omega)^2\right]}}$$

$$\cancel{= \frac{V^2/L^2 \left[\omega_0 \pm \frac{1}{2}\Delta\omega\right]^2}{\alpha^2\left(\omega_0 \pm \frac{\Delta\omega}{2}\right)^2 + \left(2\omega_0 \pm \frac{1}{2}\Delta\omega\right)^2 \frac{1}{4}(\Delta\omega)^2}}$$

$$I^2 = \frac{(V^2/L^2)(\omega_0 \pm \frac{1}{2}\Delta\omega)^2}{\left[\omega_0^2 - (\omega_0 \pm \frac{1}{2}\Delta\omega)^2\right]^2 + \alpha^2\left[\omega_0 \pm \frac{1}{2}\Delta\omega\right]^2}$$

$$= \frac{V^2/L^2 (\omega_0 \pm \frac{1}{2}\Delta\omega)^2}{\left[-\frac{(\Delta\omega)^2}{4} \mp \omega_0 \Delta\omega\right]^2 + \alpha^2\omega_0^2 \pm \alpha^2 \omega_0 \Delta\omega + \frac{\alpha^2 \Delta\omega^2}{4}}$$