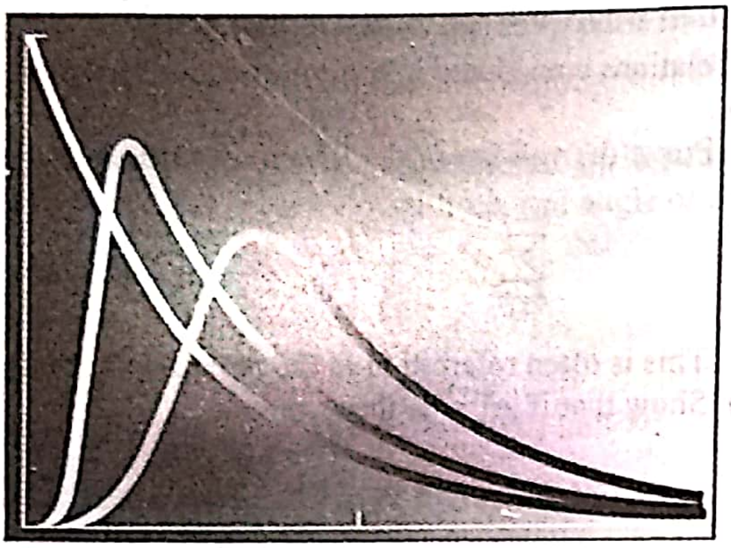
LINEAR TIME-INVARIANT SYSTEMS



2.0 INTRODUCTION

In Section 1.6 we introduced and discussed a number of basic system properties. Two of these, linearity and time invariance, play a fundamental role in signal and system analysis for two major reasons. First, many physical processes possess these properties and thus can be modeled as linear time-invariant (LTI) systems. In addition, LTI systems can be analyzed in considerable detail, providing both insight into their properties and a set of powerful tools that form the core of signal and system analysis.

A principal objective of this book is to develop an understanding of these properties and tools and to provide an introduction to several of the very important applications in which the tools are used. In this chapter, we begin the development by deriving and examining a fundamental and extremely useful representation for LTI systems and by introducing an important class of these systems.

One of the primary reasons LTI systems are amenable to analysis is that any such system possesses the superposition property described in Section 1.6.6. As a consequence, if we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.

As we will see in the following sections, one of the important characteristics of the unit impulse, both in discrete time and in continuous time, is that very general signals can be represented as linear combinations of delayed impulses. This fact, together with the properties of superposition and time invariance, will allow us to develop a complete characterization of any LTI system in terms of its response to a unit impulse. Such a representation, referred to as the convolution sum in the discrete-time case and the convolution integral in continuous time, provides considerable analytical convenience in dealing with LTI systems. Following our development of the convolution sum and the convolution integral we use these characterizations to examine some of the other properties of LTI systems. We then consider the class of continuous-time systems described by linear constant-coefficient differential equations and its discrete-time counterpart, the class of systems described by linear constant-coefficient difference equations. We will return to examine these two very important classes of systems on a number of occasions in subsequent chapters. Finally, we will take another look at the continuous-time unit impulse function and a number of other signals that are closely related to it in order to provide some additional insight into these idealized signals and, in particular, to their use and interpretation in the context of analyzing LTI systems.

2.1 DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses

The key idea in visualizing how the discrete-time unit impulse can be used to construct any discrete-time signal is to think of a discrete-time signal as a sequence of individual impulses. To see how this intuitive picture can be turned into a mathematical representation, consider the signal $x[n]$ depicted in Figure 2.1(a). In the remaining parts of this figure, we have depicted five time-shifted, scaled unit impulse sequences, where the scaling on each impulse equals the value of $x[n]$ at the particular instant the unit sample occurs. For example,

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases},$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

Therefore, the sum of the five sequences in the figure equals $x[n]$ for $-2 \leq n \leq 2$. More generally, by including additional shifted, scaled impulses, we can write

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots. \tag{2.1}$$

For any value of n, only one of the terms on the right-hand side of eq. (2.1) is nonzero, and the scaling associated with that term is precisely $x[n]$. Writing this summation in a more compact form, we have

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \tag{2.2}$$

This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulses $\delta[n-k]$, where the weights in this linear combination are $x[k]$. As an example, consider $x[n] = u[n]$, the unit step. In this case, since $u[k] = 0$ for $k < 0$

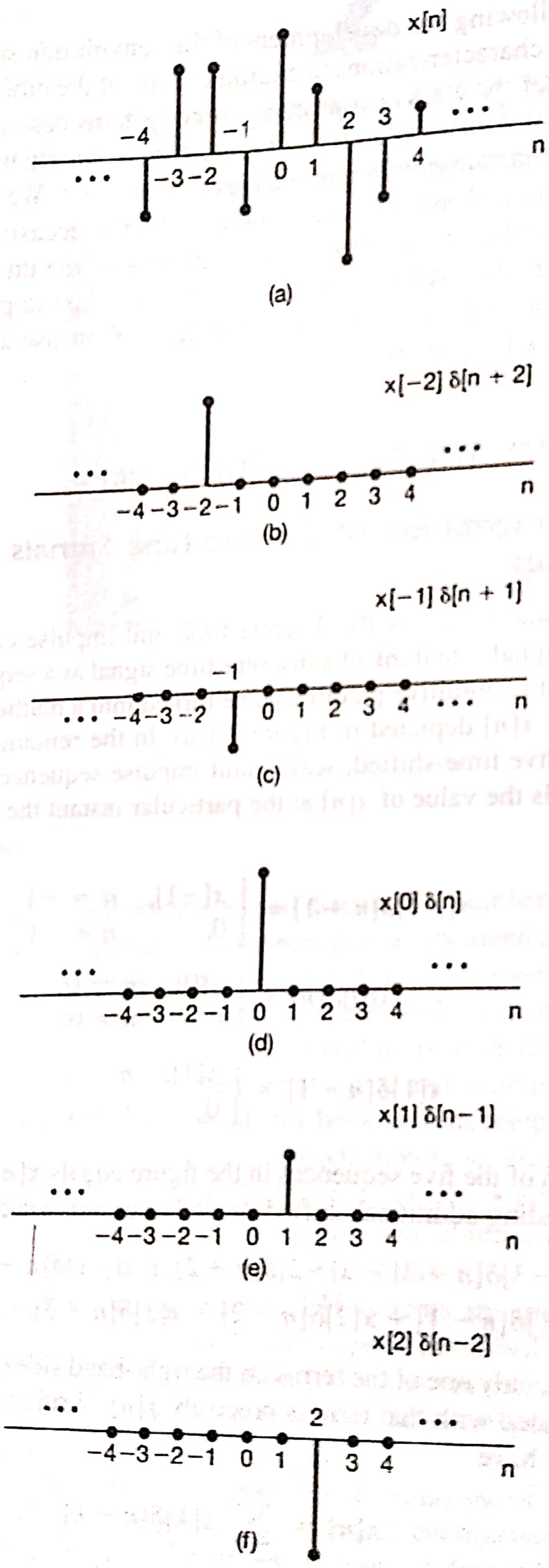


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

and $u[k] = 1$ for $k \geq 0$, eq. (2.2) becomes

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k],$$

which is identical to the expression we derived in Section 1.4. [See eq. (1.67).]

Equation (2.2) is called the *sifting property* of the discrete-time unit impulse. Because the sequence $\delta[n-k]$ is nonzero only when $k = n$, the summation on the right-hand side of eq. (2.2) "sifts" through the sequence of values $x[k]$ and preserves only the value corresponding to $k = n$. In the next subsection, we will exploit this representation of discrete-time signals in order to develop the convolution-sum representation for a discrete-time LTI system.

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

The importance of the sifting property of eqs. (2.1) and (2.2) lies in the fact that it represents $x[n]$ as a superposition of scaled versions of a very simple set of elementary functions, namely, shifted unit impulses $\delta[n-k]$, each of which is nonzero (with value 1) at a single point in time specified by the corresponding value of k. The response of a linear system to $x[n]$ will be the superposition of the scaled responses of the system to each of these shifted impulses. Moreover, the property of time invariance tells us that the responses of a time-invariant system to the time-shifted unit impulses are simply time-shifted versions of one another. The convolution-sum representation for discrete-time systems that are *both* linear and time invariant results from putting these two basic facts together.

More specifically, consider the response of a linear (but possibly time-varying) system to an arbitrary input $x[n]$. We can represent the input through eq. (2.2) as a linear combination of shifted unit impulses. Let $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-k]$. Then, from the superposition property for a linear system [eqs. (1.123) and (1.124)], the response $y[n]$ of the linear system to the input $x[n]$ in eq. (2.2) is simply the weighted linear combination of these basic responses. That is, with the input $x[n]$ to a linear system expressed in the form of eq. (2.2), the output $y[n]$ can be expressed as

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]. \quad (2.3)$$

Thus, according to eq. (2.3), if we know the response of a linear system to the set of shifted unit impulses, we can construct the response to an arbitrary input. An interpretation of eq. (2.3) is illustrated in Figure 2.2. The signal $x[n]$ is applied as the input to a linear system whose responses $h_{-1}[n]$, $h_0[n]$, and $h_1[n]$ to the signals $\delta[n+1]$, $\delta[n]$, and $\delta[n-1]$, respectively, are depicted in Figure 2.2(b). Since $x[n]$ can be written as a linear combination of $\delta[n+1]$, $\delta[n]$, and $\delta[n-1]$, superposition allows us to write the response to $x[n]$ as a linear combination of the responses to the individual shifted impulses. The individual shifted and scaled impulses that constitute $x[n]$ are illustrated on the left-hand side of Figure 2.2(c), while the responses to these component signals are pictured on the right-hand side. In Figure 2.2(d) we have depicted the actual input $x[n]$, which is the sum of the components on the left side of Figure 2.2(c) and the actual output $y[n]$, which, by

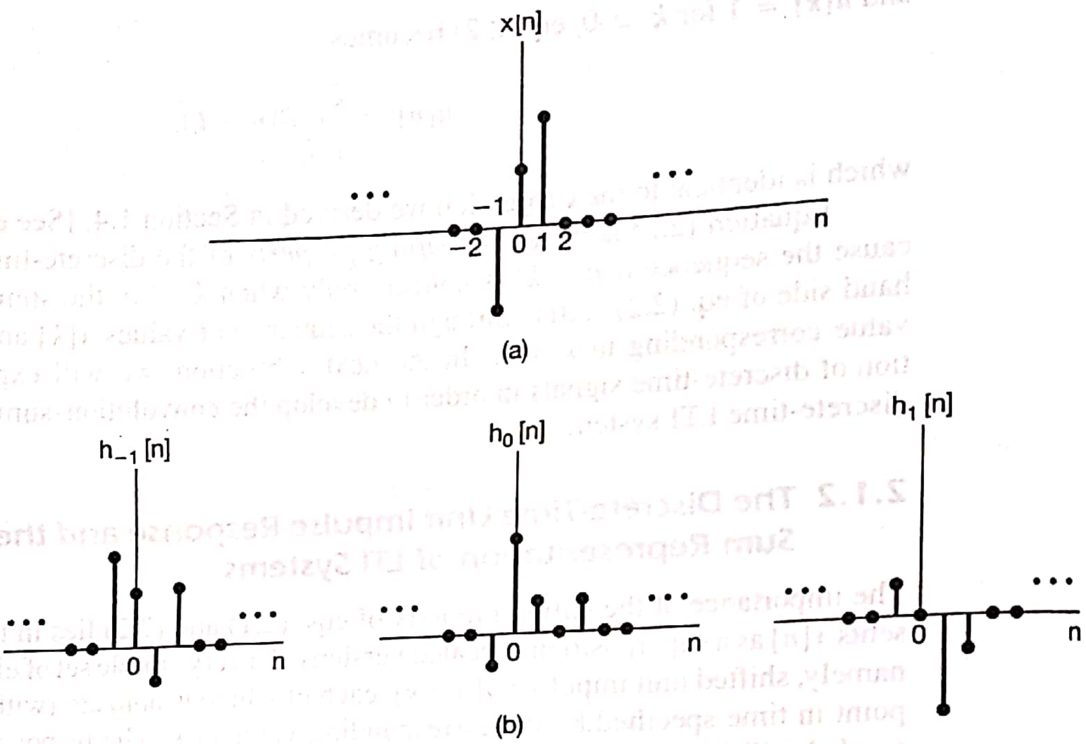


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).

superposition, is the sum of the components on the right side of Figure 2.2(c). Thus, the response at time n of a linear system is simply the superposition of the responses due to the input value at each point in time.

In general, of course, the responses $h_k[n]$ need not be related to each other for different values of k. However, if the linear system is also *time invariant*, then these responses to time-shifted unit impulses are all time-shifted versions of each other. Specifically, since $\delta[n-k]$ is a time-shifted version of $\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h_0[n]$; i.e.,

$$h_k[n] = h_0[n-k]. \tag{2.4}$$

For notational convenience, we will drop the subscript on $h_0[n]$ and define the *unit impulse (sample) response*

$$h[n] = h_0[n]. \tag{2.5}$$

That is, $h[n]$ is the output of the LTI system when $\delta[n]$ is the input. Then for an LTI system, eq. (2.3) becomes

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].} \tag{2.6}$$

This result is referred to as the *convolution sum* or *superposition sum*, and the operation on the right-hand side of eq. (2.6) is known as the *convolution* of the sequences $x[n]$ and $h[n]$. We will represent the operation of convolution symbolically as

$$y[n] = x[n] * h[n]. \tag{2.7}$$

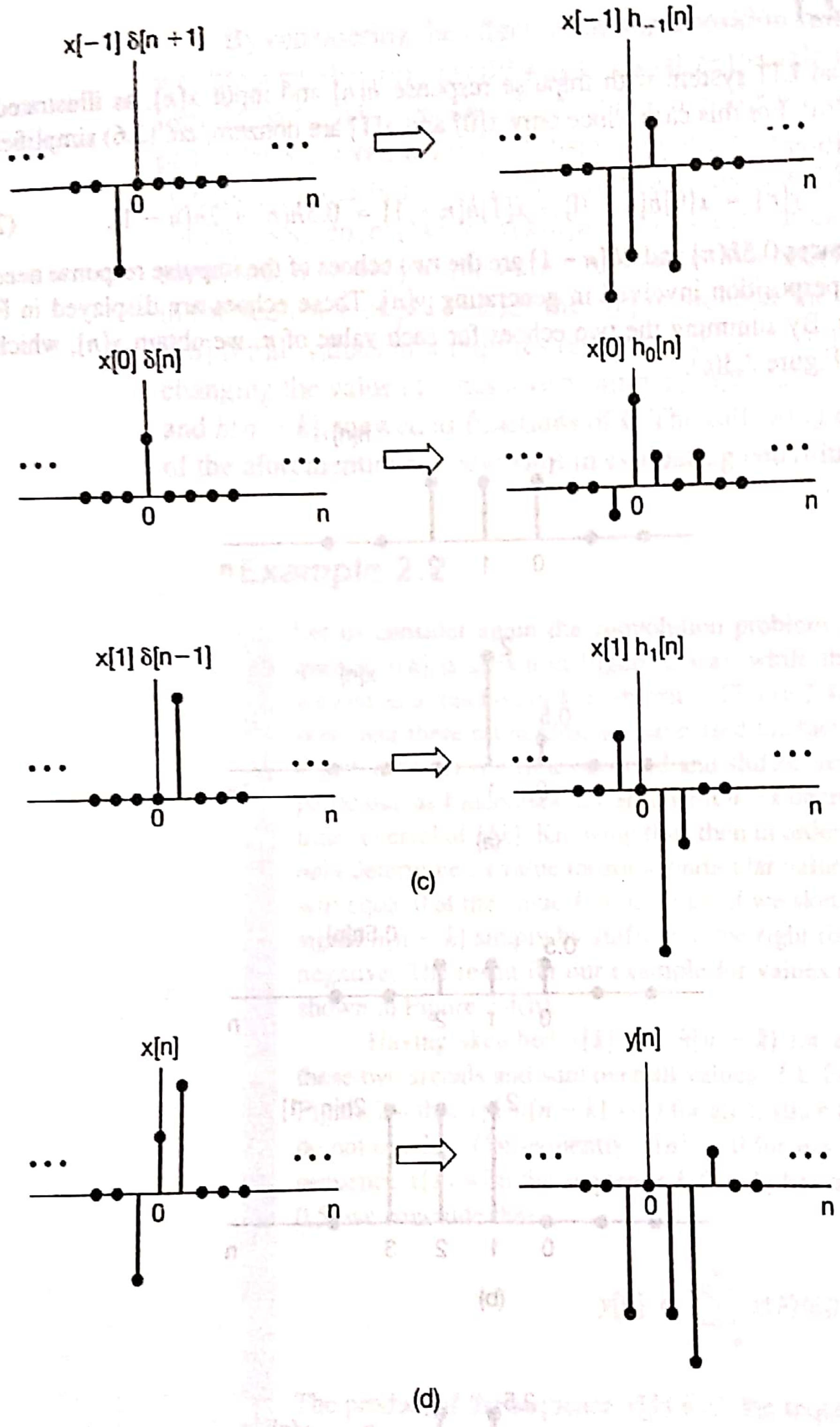


Figure 2.2 *Continued*

Note that eq. (2.6) expresses the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse. From this, we see that an LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.

The interpretation of eq. (2.6) is similar to the one we gave for eq. (2.3), where, in the case of an LTI system, the response due to the input $x[k]$ applied at time k is $x[k]h[n-k]$; i.e., it is a shifted and scaled version (an "echo") of $h[n]$. As before, the actual output is the superposition of all these responses.

Example 2.1

Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated in Figure 2.3(a). For this case, since only $x[0]$ and $x[1]$ are nonzero, eq. (2.6) simplifies to the expression

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]. \tag{2.8}$$

The sequences $0.5h[n]$ and $2h[n-1]$ are the two echoes of the impulse response needed for the superposition involved in generating $y[n]$. These echoes are displayed in Figure 2.3(b). By summing the two echoes for each value of n, we obtain $y[n]$, which is shown in Figure 2.3(c).

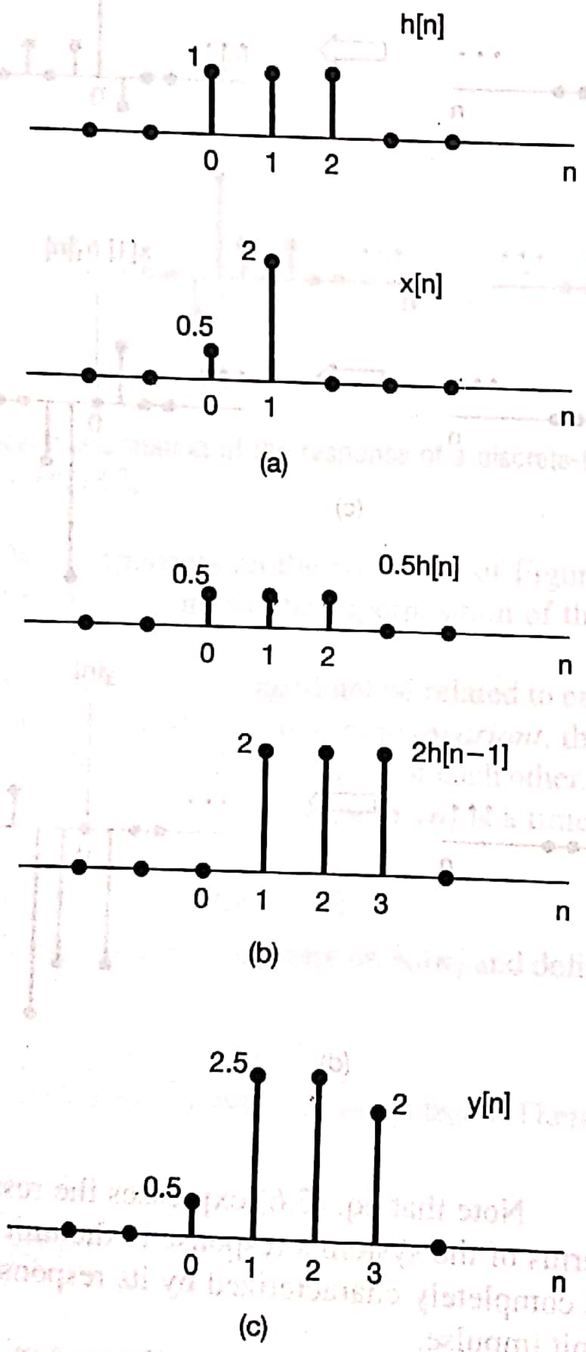


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

By considering the effect of the superposition sum on each individual output sample, we obtain another very useful way to visualize the calculation of $y[n]$ using the convolution sum. In particular, consider the evaluation of the output value at some specific time n. A particularly convenient way of displaying this calculation graphically begins with the two signals $x[k]$ and $h[n-k]$ viewed as functions of k. Multiplying these two functions, we obtain a sequence $g[k] = x[k]h[n-k]$, which, at each time k, is seen to represent the contribution of $x[k]$ to the output at time n. We conclude that summing all the samples in the sequence of $g[k]$ yields the output value at the selected time n. Thus, to calculate $y[n]$ for all values of n requires repeating this procedure for each value of n. Fortunately, changing the value of n has a very simple graphical interpretation for the two signals $x[k]$ and $h[n-k]$, viewed as functions of k. The following examples illustrate this and the use of the aforementioned viewpoint in evaluating convolution sums.

Example 2.2

Let us consider again the convolution problem encountered in Example 2.1. The sequence $x[k]$ is shown in Figure 2.4(a), while the sequence $h[n-k]$, for n fixed and viewed as a function of k, is shown in Figure 2.4(b) for several different values of n. In sketching these sequences, we have used the fact that $h[n-k]$ (viewed as a function of k with n fixed) is a time-reversed and shifted version of the impulse response $h[k]$. In particular, as k increases, the argument $n-k$ decreases, explaining the need to perform a time reversal of $h[k]$. Knowing this, then in order to sketch the signal $h[n-k]$, we need only determine its value for some particular value of k. For example, the argument $n-k$ will equal 0 at the value $k = n$. Thus, if we sketch the signal $h[-k]$, we can obtain the signal $h[n-k]$ simply by shifting to the right (by n) if n is positive or to the left if n is negative. The result for our example for values of $n < 0$, $n = 0, 1, 2, 3$, and $n > 3$ are shown in Figure 2.4(b).

Having sketched $x[k]$ and $h[n-k]$ for any particular value of n, we multiply these two signals and sum over all values of k. For our example, for $n < 0$, we see from Figure 2.4 that $x[k]h[n-k] = 0$ for all k, since the nonzero values of $x[k]$ and $h[n-k]$ do not overlap. Consequently, $y[n] = 0$ for $n < 0$. For $n = 0$, since the product of the sequence $x[k]$ with the sequence $h[0-k]$ has only one nonzero sample with the value 0.5, we conclude that

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5. \tag{2.9}$$

The product of the sequence $x[k]$ with the sequence $h[1-k]$ has two nonzero samples, which may be summed to obtain

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5. \tag{2.10}$$

Similarly,

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5, \tag{2.11}$$

and

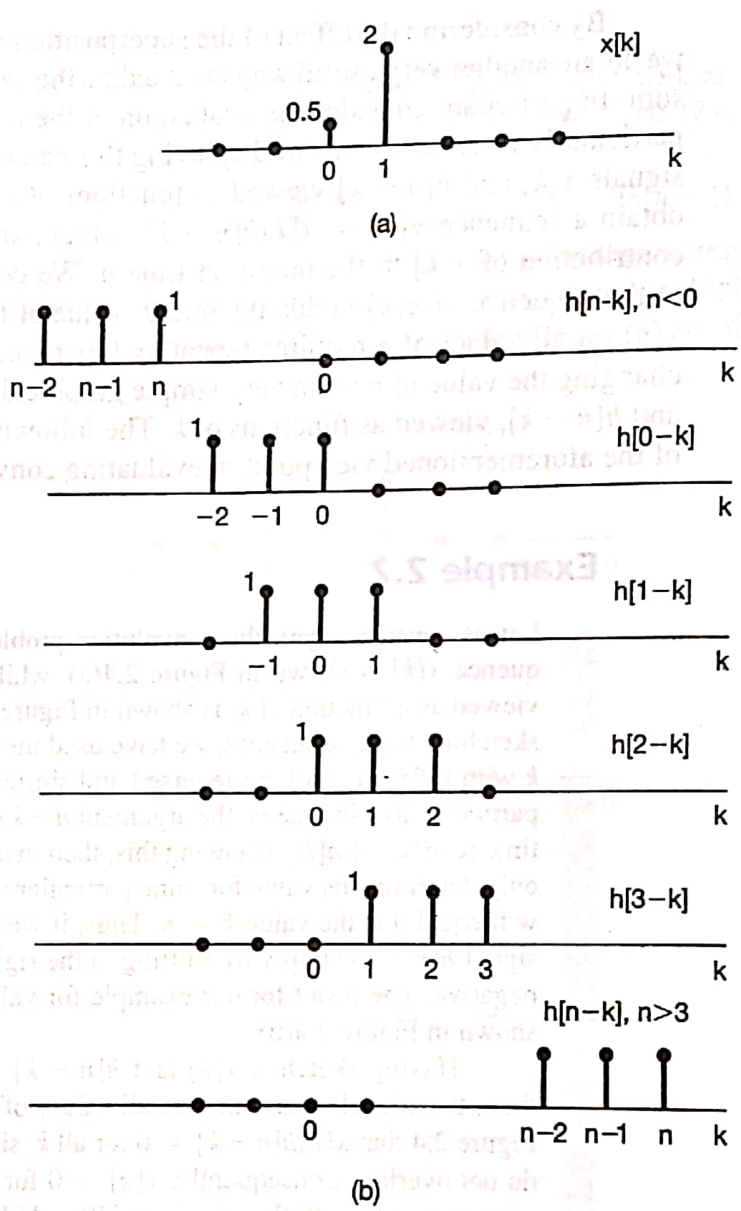


Figure 2.4 Interpretation of eq. (2.6) for the signals $h[n]$ and $x[n]$ in Figure 2.3; (a) the signal $x[k]$ and (b) the signal $h[n-k]$ (as a function of k with n fixed) for several values of n ($n < 0$; $n = 0, 1, 2, 3$; $n > 3$). Each of these signals is obtained by reflection and shifting of the unit impulse response $h[k]$. The response $y[n]$ for each value of n is obtained by multiplying the signals $x[k]$ and $h[n-k]$ in (b) and (c) and then summing the products over all values of k. The calculation for this example is carried out in detail in Example 2.2.

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 2.0. \tag{2.12}$$

Finally, for $n > 3$, the product $x[k]h[n-k]$ is zero for all k, from which we conclude that $y[n] = 0$ for $n > 3$. The resulting output values agree with those obtained in Example 2.1.

Example 2.3

Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \alpha^n u[n],$$
$$h[n] = u[n],$$

with $0 < \alpha < 1$. These signals are illustrated in Figure 2.5. Also, to help us in visualizing and calculating the convolution of the signals, in Figure 2.6 we have depicted the signal $x[k]$ followed by $h[-k]$, $h[-1-k]$, and $h[1-k]$ (that is, $h[n-k]$ for $n = 0, -1$, and $+1$) and, finally, $h[n-k]$ for an arbitrary positive value of n and an arbitrary negative value of n. From this figure, we note that for $n < 0$, there is no overlap between the nonzero points in $x[k]$ and $h[n-k]$. Thus, for $n < 0$, $x[k]h[n-k] = 0$ for all values of k, and hence, from eq. (2.6), we see that $y[n] = 0, n < 0$. For $n \geq 0$,

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

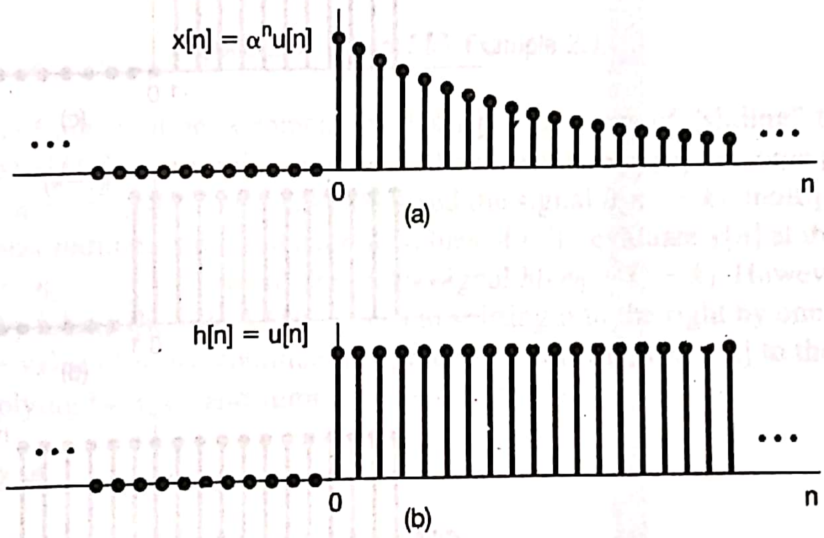


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

Thus, for $n \geq 0$,

$$y[n] = \sum_{k=0}^{n} \alpha^k,$$

and using the result of Problem 1.54 we can write this as

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{for } n \geq 0. \tag{2.13}$$

Thus, for all n,

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n].$$

The signal $y[n]$ is sketched in Figure 2.7.

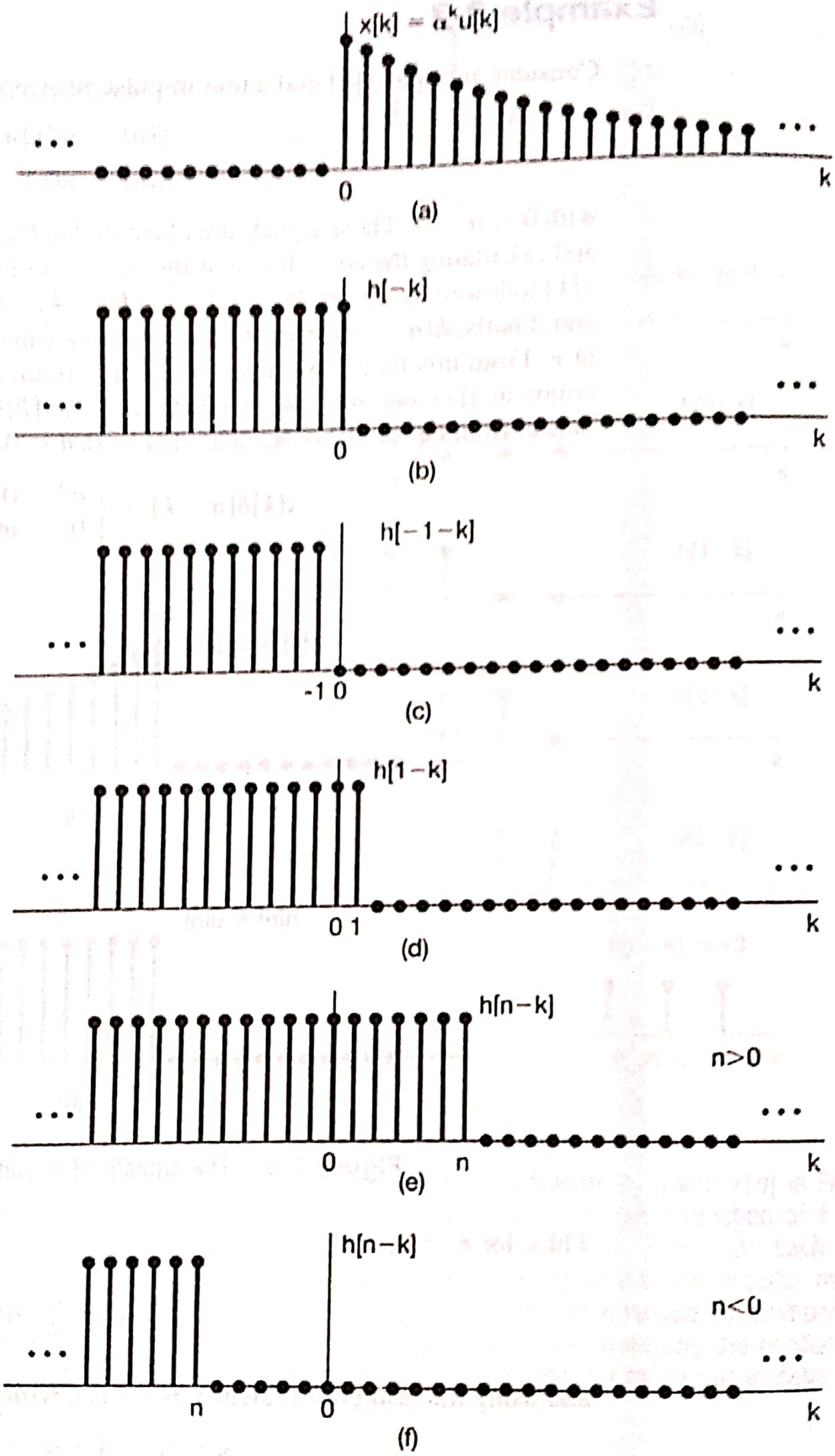


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

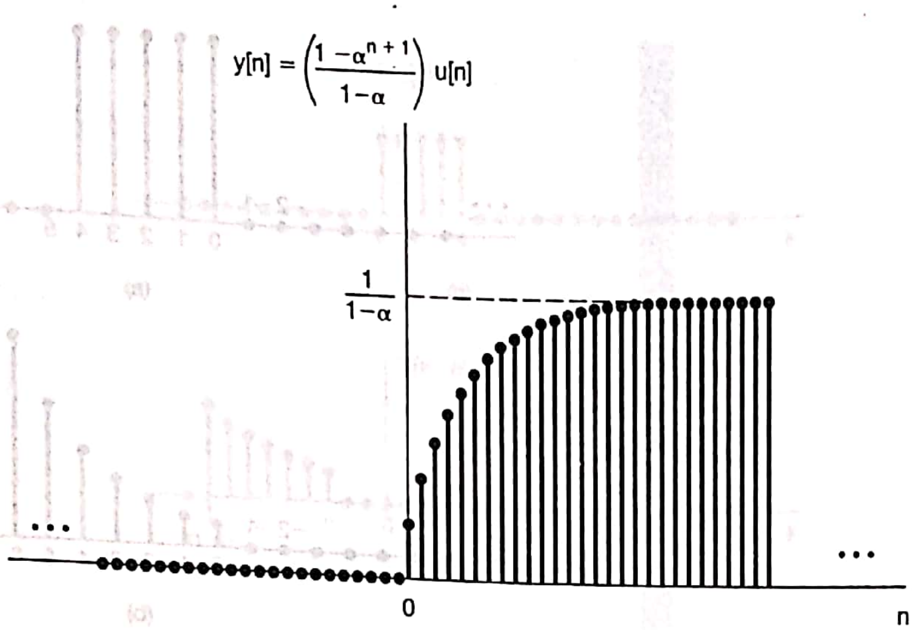


Figure 2.7 Output for Example 2.3.

The operation of convolution is sometimes described in terms of "sliding" the sequence $h[n-k]$ past $x[k]$. For example, suppose we have evaluated $y[n]$ for some particular value of n, say, $n = n_0$. That is, we have sketched the signal $h[n_0 - k]$, multiplied it by the signal $x[k]$, and summed the result over all values of k. To evaluate $y[n]$ at the next value of n—i.e., $n = n_0 + 1$—we need to sketch the signal $h[(n_0 + 1) - k]$. However, we can do this simply by taking the signal $h[n_0 - k]$ and shifting it to the right by one point. For each successive value of n, we continue this process of shifting $h[n-k]$ to the right by one point, multiplying by $x[k]$, and summing the result over k.

Example 2.4

As a further example, consider the two sequences

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

These signals are depicted in Figure 2.8 for a positive value of $\alpha > 1$. In order to calculate the convolution of the two signals, it is convenient to consider five separate intervals for n. This is illustrated in Figure 2.9.

Interval 1. For $n < 0$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and consequently, $y[n] = 0$.

Interval 2. For $0 \le n \le 4$,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}.$$

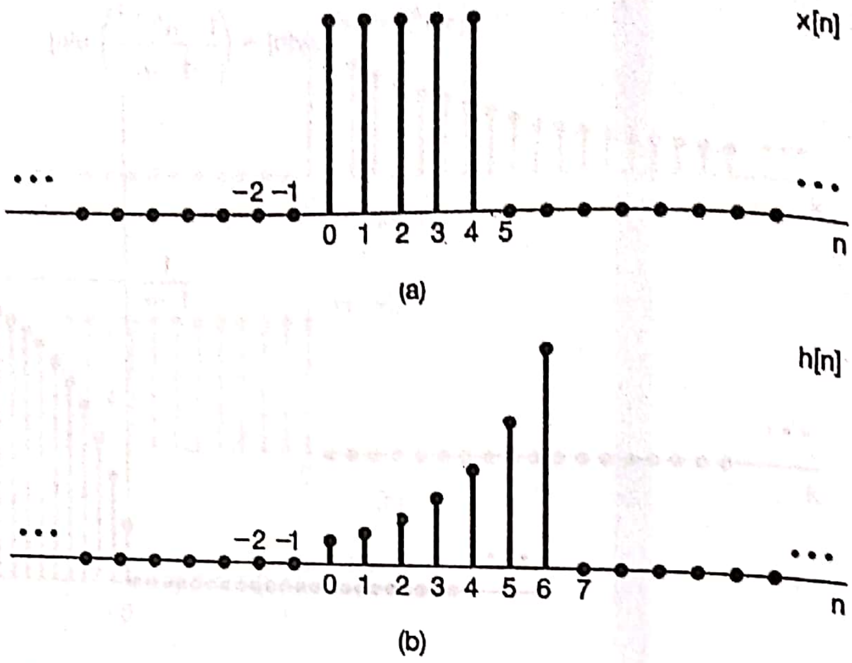


Figure 2.8 The signals to be convolved in Example 2.4.

Thus, in this interval,

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k} \tag{2.14}$$

We can evaluate this sum using the finite sum formula, eq. (2.13). Specifically, changing the variable of summation in eq. (2.14) from k to $r = n - k$, we obtain

$$y[n] = \sum_{r=0}^{n} \alpha^{r} = \frac{1-\alpha^{n+1}}{1-\alpha}.$$

Interval 3. For $n > 4$ but $n - 6 \leq 0$ (i.e., $4 < n \leq 6$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

Thus, in this interval,

$$y[n] = \sum_{k=0}^{4} \alpha^{n-k}. \tag{2.15}$$

Once again, we can use the geometric sum formula in eq. (2.13) to evaluate eq. (2.15). Specifically, factoring out the constant factor of α^n from the summation in eq. (2.15) yields

$$y[n] = \alpha^n \sum_{k=0}^{4} (\alpha^{-1})^k = \alpha^n \frac{1-(\alpha^{-1})^5}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}. \tag{2.16}$$

Interval 4. For $n > 6$ but $n - 6 \leq 4$ (i.e., for $6 < n \leq 10$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases},$$

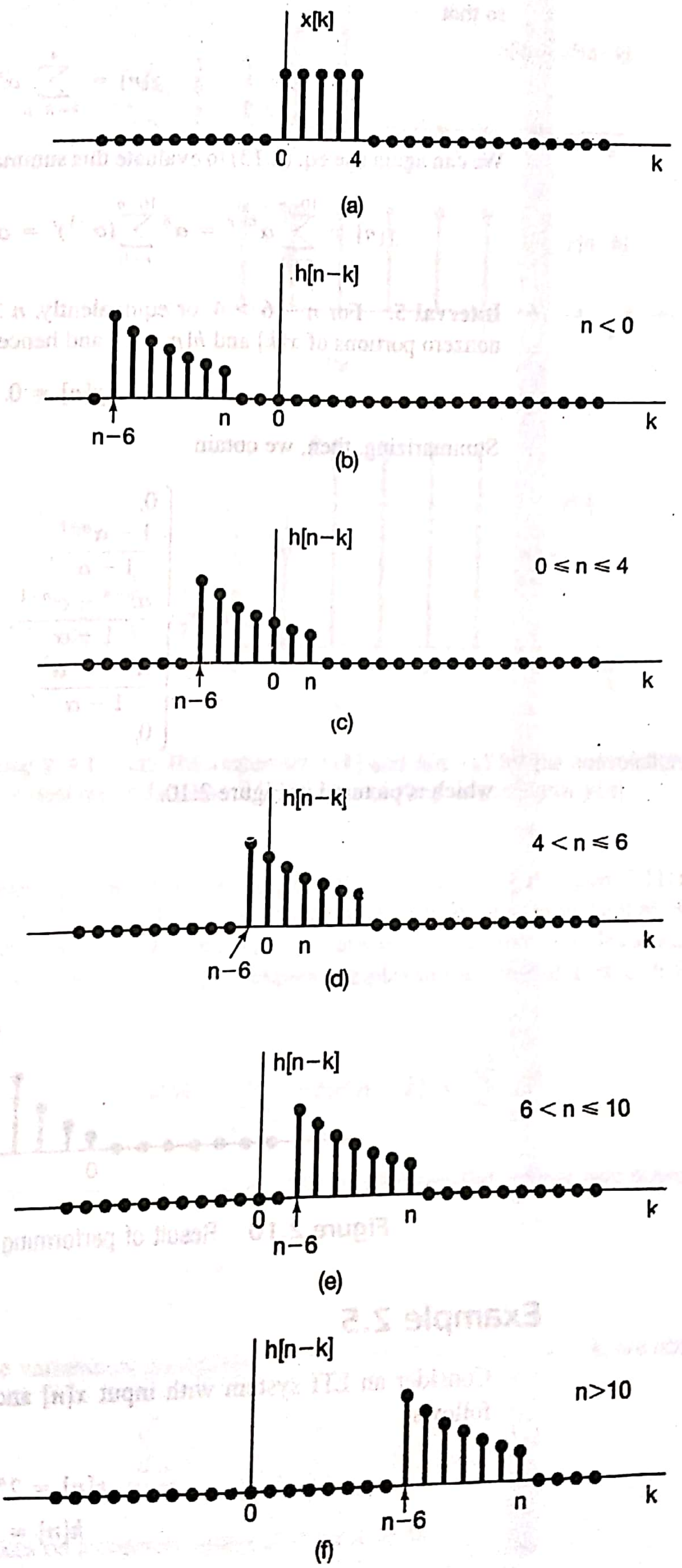


Figure 2.9 Graphical interpretation of the convolution performed in Example 2.4.

so that

$$y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}$$

We can again use eq. (2.13) to evaluate this summation. Letting $r = k-n+6$, we obtain

$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^6 \sum_{r=0}^{10-n} (\alpha^{-1})^r = \alpha^6 \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^7}{1-\alpha}.$$

Interval 5. For $n-6>4$, or equivalently, $n>10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and hence,

$$y[n] = 0.$$

Summarizing, then, we obtain

$$y[n] = \begin{cases} 0, & n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4<n\le 6 \\ \dfrac{\alpha^{n-4}-\alpha^7}{1-\alpha}, & 6<n\le 10 \\ 0, & 10<n \end{cases}$$

which is pictured in Figure 2.10.

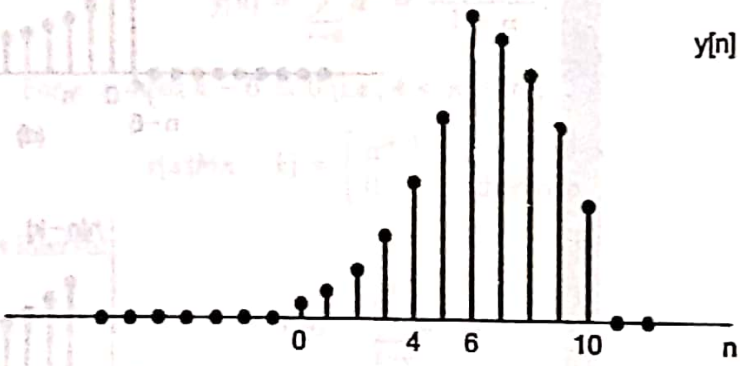


Figure 2.10 Result of performing the convolution in Example 2.4.

Example 2.5

Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows:

$$x[n] = 2^n u[-n], \quad (2.17)$$

$$h[n] = u[n]. \quad (2.18)$$

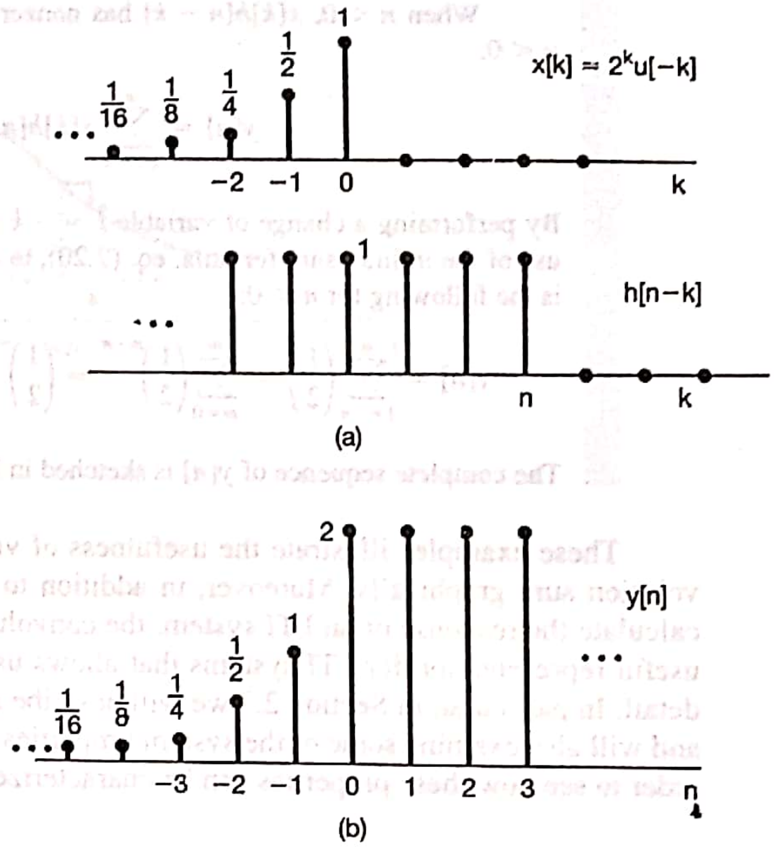


Figure 2.11 (a) The sequences $x[k]$ and $h[n-k]$ for the convolution problem considered in Example 2.5; (b) the resulting output signal $y[n]$.

The sequences $x[k]$ and $h[n-k]$ are plotted as functions of k in Figure 2.11(a). Note that $x[k]$ is zero for $k > 0$ and $h[n-k]$ is zero for $k > n$. We also observe that, regardless of the value of n, the sequence $x[k]h[n-k]$ always has nonzero samples along the k-axis. When $n \geq 0$, $x[k]h[n-k]$ has nonzero samples in the interval $k \leq 0$. It follows that, for $n \geq 0$,

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k. \tag{2.19}$$

To evaluate the infinite sum in eq. (2.19), we may use the *infinite sum formula*,

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \quad 0 < |\alpha| < 1. \tag{2.20}$$

Changing the variable of summation in eq. (2.19) from k to $r = -k$, we obtain

$$\sum_{k=-\infty}^{0} 2^k = \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-(1/2)} = 2. \tag{2.21}$$

Thus, $y[n]$ takes on a constant value of 2 for $n \geq 0$.

When $n < 0$, $x[k]h[n-k]$ has nonzero samples for $k \leq n$. It follows that, for $n < 0$,

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{n} 2^k. \tag{2.22}$$

By performing a change of variable $l = -k$ and then $m = l + n$, we can again make use of the infinite sum formula, eq. (2.20), to evaluate the sum in eq. (2.22). The result is the following for $n < 0$:

$$y[n] = \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n} = \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}. \tag{2.23}$$

The complete sequence of $y[n]$ is sketched in Figure 2.11(b).

These examples illustrate the usefulness of visualizing the calculation of the convolution sum graphically. Moreover, in addition to providing a useful way in which to calculate the response of an LTI system, the convolution sum also provides an extremely useful representation for LTI systems that allows us to examine their properties in great detail. In particular, in Section 2.3 we will describe some of the properties of convolution and will also examine some of the system properties introduced in the previous chapter in order to see how these properties can be characterized for LTI systems.

2.2 CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

In analogy with the results derived and discussed in the preceding section, the goal of this section is to obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response. In discrete time, the key to our developing the convolution sum was the sifting property of the discrete-time unit impulse—that is, the mathematical representation of a signal as the superposition of scaled and shifted unit impulse functions. Intuitively, then, we can think of the discrete-time system as responding to a sequence of individual impulses. In continuous time, of course, we do not have a discrete sequence of input values. Nevertheless, as we discussed in Section 1.4.2, if we think of the unit impulse as the idealization of a pulse which is so short that its duration is inconsequential for any real, physical system, we can develop a representation for arbitrary continuous-time signals in terms of these idealized pulses with vanishingly small duration, or equivalently, impulses. This representation is developed in the next subsection, and, following that, we will proceed very much as in Section 2.1 to develop the convolution integral representation for continuous-time LTI systems.

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

To develop the continuous-time counterpart of the discrete-time sifting property in eq. (2.2), we begin by considering a pulse or "staircase" approximation, $\hat{x}(t)$, to a continuous-time signal $x(t)$, as illustrated in Figure 2.12(a). In a manner similar to that

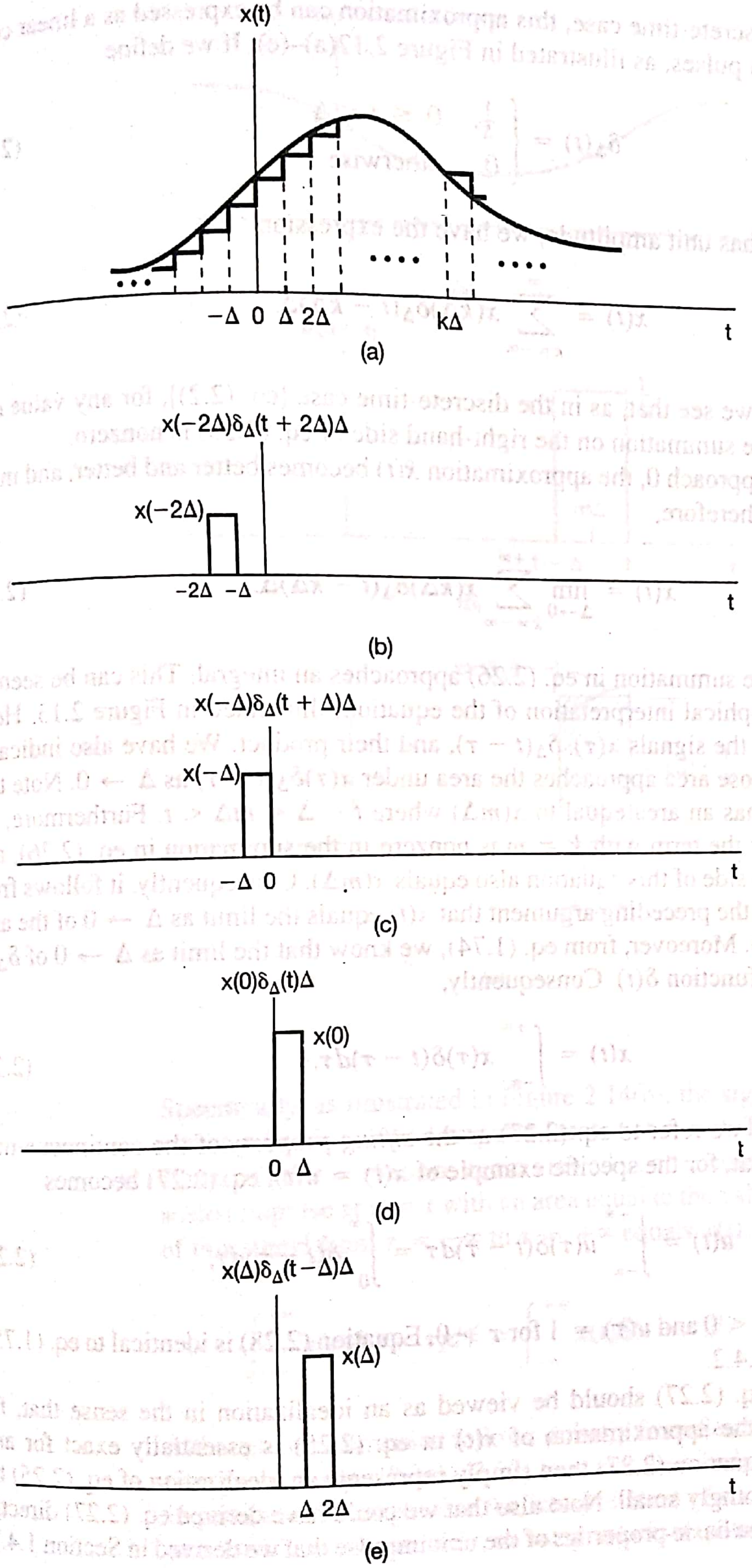


Figure 2.12 Staircase approximation to a continuous-time signal.

employed in the discrete-time case, this approximation can be expressed as a linear combination of delayed pulses, as illustrated in Figure 2.12(a)–(e). If we define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}, \tag{2.24}$$

then, since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \tag{2.25}$$

From Figure 2.12, we see that, as in the discrete-time case [eq. (2.2)], for any value of t, only one term in the summation on the right-hand side of eq. (2.25) is nonzero.

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals $x(t)$. Therefore,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \tag{2.26}$$

Also, as $\Delta \to 0$, the summation in eq. (2.26) approaches an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure 2.13. Here, we have illustrated the signals $x(\tau)$, $\delta_\Delta(t - \tau)$, and their product. We have also indicated a shaded region whose area approaches the area under $x(\tau)\delta_\Delta(t - \tau)$ as $\Delta \to 0$. Note that the shaded region has an area equal to $x(m\Delta)$ where $t - \Delta < m\Delta < t$. Furthermore, for this value of t, only the term with $k = m$ is nonzero in the summation in eq. (2.26), and thus, the right-hand side of this equation also equals $x(m\Delta)$. Consequently, it follows from eq. (2.26) and from the preceding argument that $x(t)$ equals the limit as $\Delta \to 0$ of the area under $x(\tau)\delta_\Delta(t - \tau)$. Moreover, from eq. (1.74), we know that the limit as $\Delta \to 0$ of $\delta_\Delta(t)$ is the unit impulse function $\delta(t)$. Consequently,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau. \tag{2.27}$$

As in discrete time, we refer to eq. (2.27) as the *sifting property* of the continuous-time impulse. We note that, for the specific example of $x(t) = u(t)$, eq. (2.27) becomes

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t - \tau)d\tau = \int_{0}^{\infty} \delta(t - \tau)d\tau, \tag{2.28}$$

since $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau > 0$. Equation (2.28) is identical to eq. (1.75), derived in Section 1.4.2.

Once again, eq. (2.27) should be viewed as an idealization in the sense that, for Δ "small enough," the approximation of $x(t)$ in eq. (2.25) is essentially exact for any practical purpose. Equation (2.27) then simply represents an idealization of eq. (2.25) by taking Δ to be vanishingly small. Note also that we could have derived eq. (2.27) directly by using several of the basic properties of the unit impulse that we derived in Section 1.4.2.

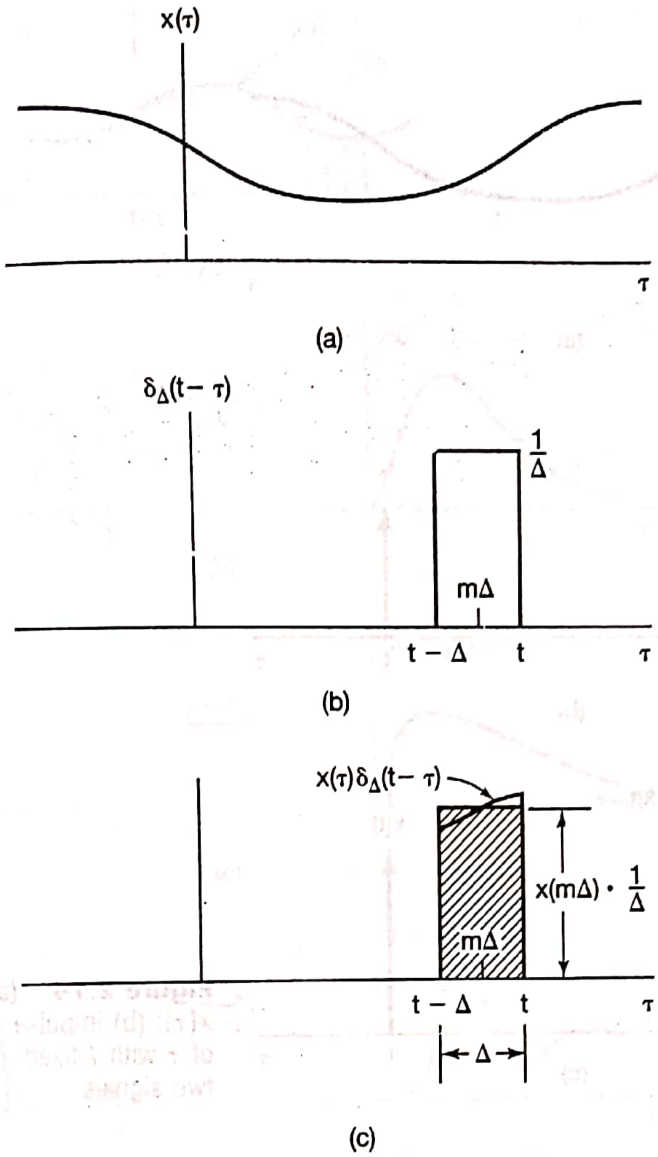


Figure 2.13 Graphical interpretation of eq. (2.26).

Specifically, as illustrated in Figure 2.14(b), the signal $\delta(t-\tau)$ (viewed as a function of τ with t fixed) is a unit impulse located at $\tau = t$. Thus, as shown in Figure 2.14(c), the signal $x(\tau)\delta(t-\tau)$ (once again viewed as a function of τ) equals $x(t)\delta(t-\tau)$ [i.e., it is a scaled impulse at $\tau = t$ with an area equal to the value of $x(t)$]. Consequently, the integral of this signal from $\tau = -\infty$ to $\tau = +\infty$ equals $x(t)$; that is,

$$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{+\infty} \delta(t-\tau)d\tau = x(t).$$

Although this derivation follows directly from Section 1.4.2, we have included the derivation given in eqs. (2.24)–(2.27) to stress the similarities with the discrete-time case and, in particular, to emphasize the interpretation of eq. (2.27) as representing the signal $x(t)$ as a "sum" (more precisely, an integral) of weighted, shifted impulses.

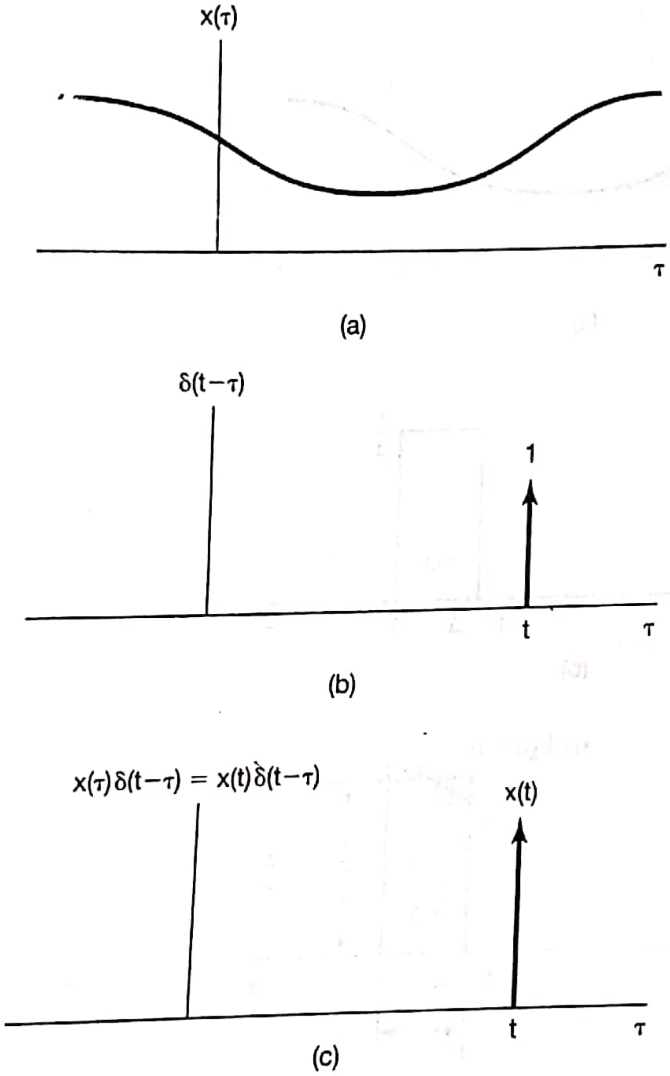


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

As in the discrete-time case, the representation developed in the preceding section provides us with a way in which to view an arbitrary continuous-time signal as the superposition of scaled and shifted pulses. In particular, the approximate representation in eq. (2.25) represents the signal $\hat{x}(t)$ as a sum of scaled and shifted versions of the basic pulse signal $\delta_\Delta(t)$. Consequently, the response $\hat{y}(t)$ of a linear system to this signal will be the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$. Specifically, let us define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$. Then, from eq. (2.25) and the superposition property, for continuous-time linear systems, we see that

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta. \tag{2.29}$$

The interpretation of eq. (2.29) is similar to that for eq. (2.3) in discrete time. In particular, consider Figure 2.15, which is the continuous-time counterpart of Figure 2.2. In

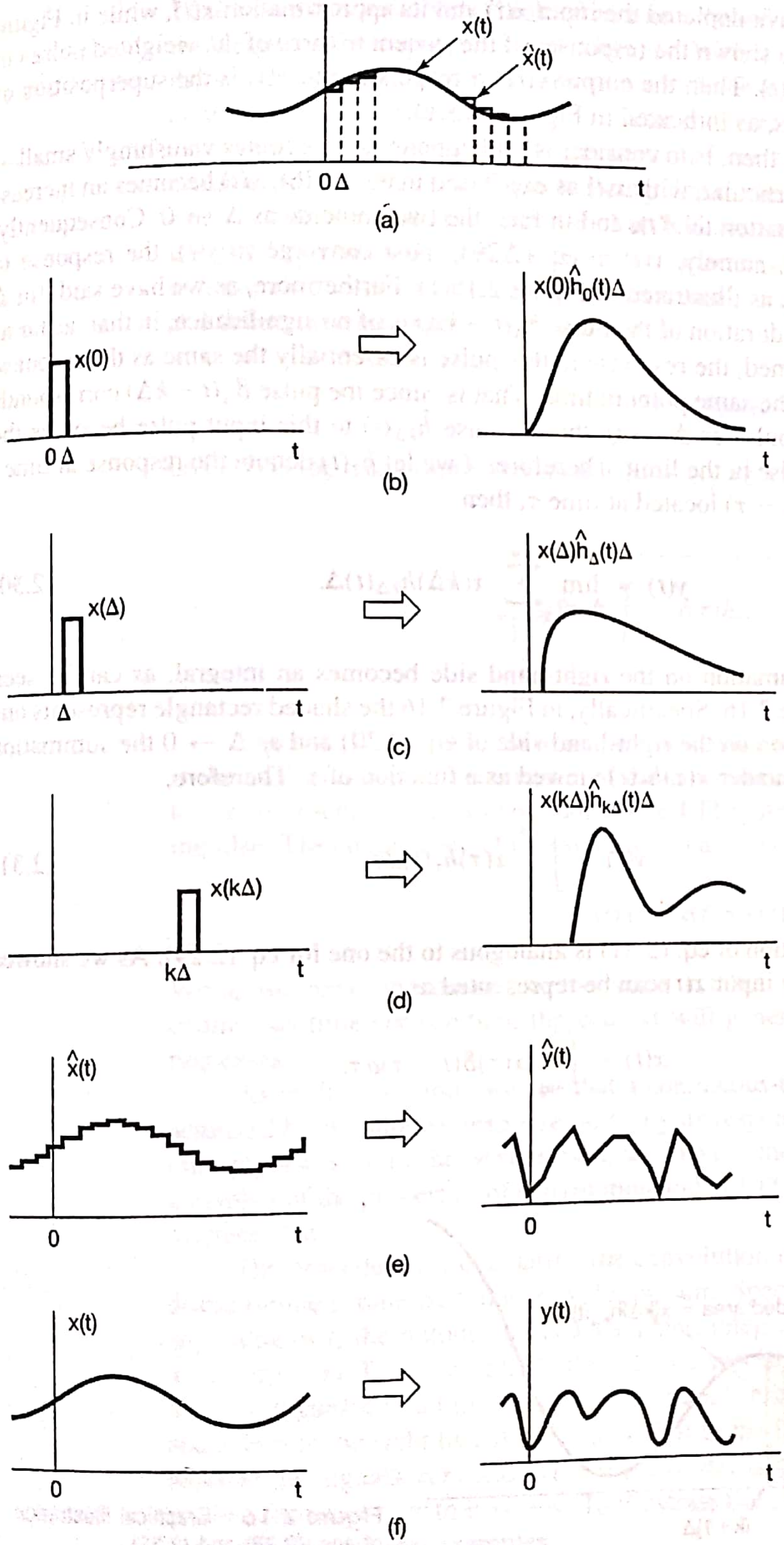


Figure 2.15 Graphical interpretation of the response of a continuous-time linear system as expressed in eqs. (2.29) and (2.30).

Figure 2.15(a) we have depicted the input $x(t)$ and its approximation $\hat{x}(t)$, while in Figure 2.15(b)–(d), we have shown the responses of the system to three of the weighted pulses in the expression for $\hat{x}(t)$. Then the output $\hat{y}(t)$ corresponding to $\hat{x}(t)$ is the superposition of all of these responses, as indicated in Figure 2.15(e).

What remains, then, is to consider what happens as Δ becomes vanishingly small—i.e., as $\Delta \rightarrow 0$. In particular, with $x(t)$ as expressed in eq. (2.26), $\hat{x}(t)$ becomes an increasingly good approximation to $x(t)$, and in fact, the two coincide as $\Delta \rightarrow 0$. Consequently, the response to $\hat{x}(t)$, namely, $\hat{y}(t)$ in eq. (2.29), must converge to $y(t)$, the response to the actual input $x(t)$, as illustrated in Figure 2.15(f). Furthermore, as we have said, for Δ "small enough," the duration of the pulse $\delta_\Delta(t - k\Delta)$ is of no significance, in that, as far as the system is concerned, the response to this pulse is essentially the same as the response to a unit impulse at the same point in time. That is, since the pulse $\delta_\Delta(t - k\Delta)$ corresponds to a shifted unit impulse as $\Delta \rightarrow 0$, the response $\hat{h}_{k\Delta}(t)$ to this input pulse becomes the response to an impulse in the limit. Therefore, if we let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t - \tau)$ located at time τ, then

$$y(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta) h_{k\Delta}(t)\Delta. \tag{2.30}$$

As $\Delta \rightarrow 0$, the summation on the right-hand side becomes an integral, as can be seen graphically in Figure 2.16. Specifically, in Figure 2.16 the shaded rectangle represents one term in the summation on the right-hand side of eq. (2.30) and as $\Delta \rightarrow 0$ the summation approaches the area under $x(\tau)h_\tau(t)$ viewed as a function of τ. Therefore,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h_\tau(t) d\tau. \tag{2.31}$$

The interpretation of eq. (2.31) is analogous to the one for eq. (2.29). As we showed in Section 2.2.1, any input $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau) d\tau.$$

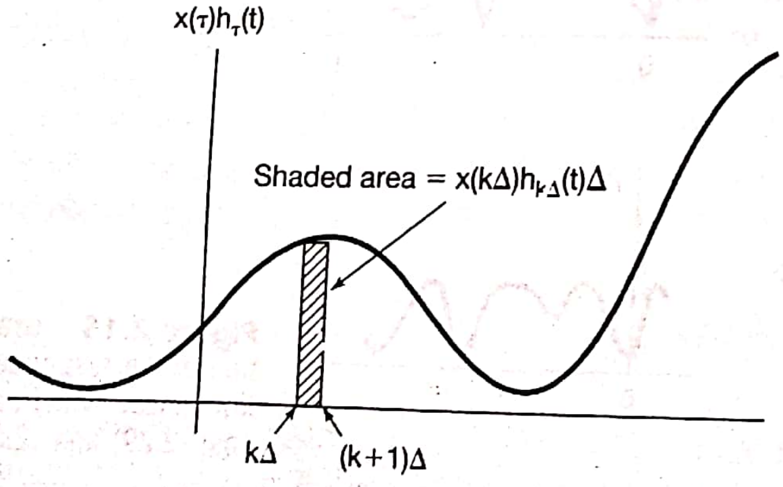


Figure 2.16 Graphical illustration of eqs. (2.30) and (2.31).

That is, we can intuitively think of $x(t)$ as a "sum" of weighted shifted impulses, where the weight on the impulse $\delta(t-\tau)$ is $x(\tau)d\tau$. With this interpretation, eq. (2.31) represents the superposition of the responses to each of these inputs, and by linearity, the weight on the response $h_\tau(t)$ to the shifted impulse $\delta(t-\tau)$ is also $x(\tau)d\tau$.

Equation (2.31) represents the general form of the response of a linear system in continuous time. If, in addition to being linear, the system is also time invariant, then $h_\tau(t) = h_0(t-\tau)$; i.e., the response of an LTI system to the unit impulse $\delta(t-\tau)$, which is shifted by τ seconds from the origin, is a similarly shifted version of the response to the unit impulse function $\delta(t)$. Again, for notational convenience, we will drop the subscript and define the *unit impulse response* $h(t)$ as

$$h(t) = h_0(t); \tag{2.32}$$

i.e., $h(t)$ is the response to $\delta(t)$. In this case, eq. (2.31) becomes

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.} \tag{2.33}$$

Equation (2.33), referred to as the *convolution integral* or the *superposition integral*, is the continuous-time counterpart of the convolution sum of eq. (2.6) and corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse. The convolution of two signals $x(t)$ and $h(t)$ will be represented symbolically as

$$y(t) = x(t) * h(t). \tag{2.34}$$

While we have chosen to use the same symbol $*$ to denote both discrete-time and continuous-time convolution, the context will generally be sufficient to distinguish the two cases.

As in discrete time, we see that a continuous-time LTI system is completely characterized by its impulse response—i.e., by its response to a single elementary signal, the unit impulse $\delta(t)$. In the next section, we explore the implications of this as we examine a number of the properties of convolution and of LTI systems in both continuous time and discrete time.

The procedure for evaluating the convolution integral is quite similar to that for its discrete-time counterpart, the convolution sum. Specifically, in eq. (2.33) we see that, for any value of t, the output $y(t)$ is a weighted integral of the input, where the weight on $x(\tau)$ is $h(t-\tau)$. To evaluate this integral for a specific value of t, we first obtain the signal $h(t-\tau)$ (regarded as a function of τ with t fixed) from $h(\tau)$ by a reflection about the origin and a shift to the right by t if $t > 0$ or a shift to the left by $|t|$ for $t < 0$. We next multiply together the signals $x(\tau)$ and $h(t-\tau)$, and $y(t)$ is obtained by integrating the resulting product from $\tau = -\infty$ to $\tau = +\infty$. To illustrate the evaluation of the convolution integral, let us consider several examples.

Example 2.6

Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = e^{-at}u(t), \quad a > 0$$

and

$$h(t) = u(t).$$

In Figure 2.17, we have depicted the functions $h(\tau)$, $x(\tau)$, and $h(t-\tau)$ for a negative value of t and for a positive value of t. From this figure, we see that for $t < 0$, the product of $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero. For $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}.$$

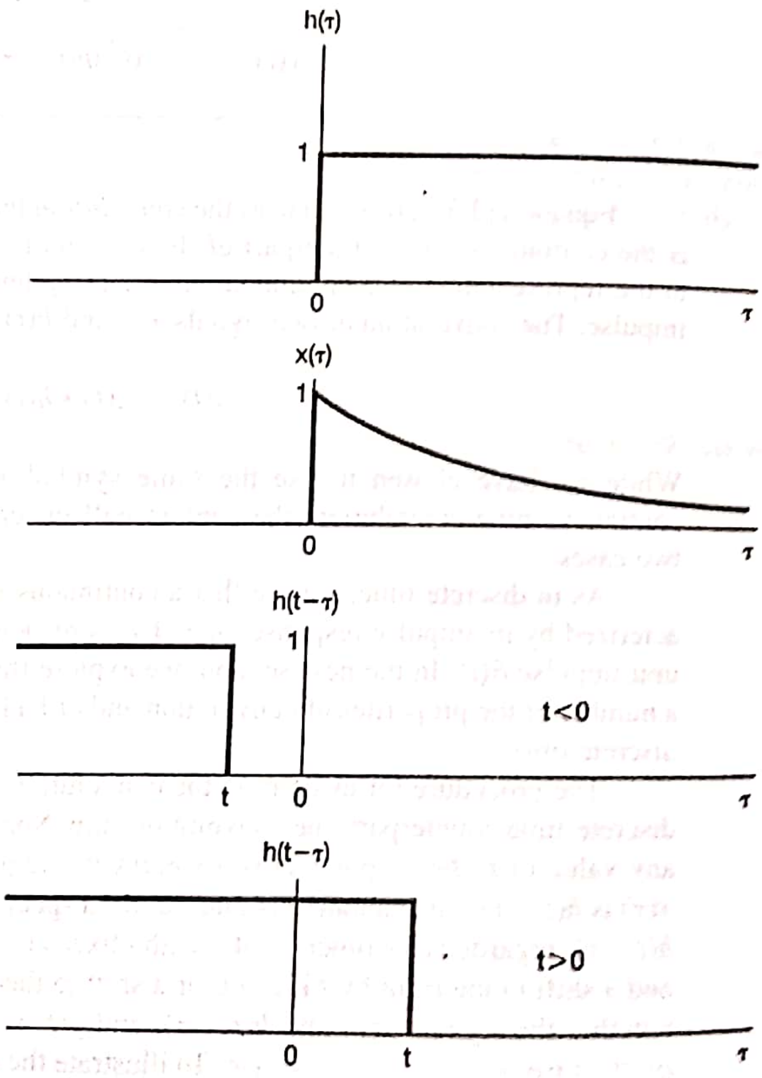


Figure 2.17 Calculation of the convolution integral for Example 2.6.

From this expression, we can compute $y(t)$ for $t > 0$:

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t$$
$$= \frac{1}{a}(1 - e^{-at}).$$

Thus, for all t, $y(t)$ is

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t),$$

which is shown in Figure 2.18.

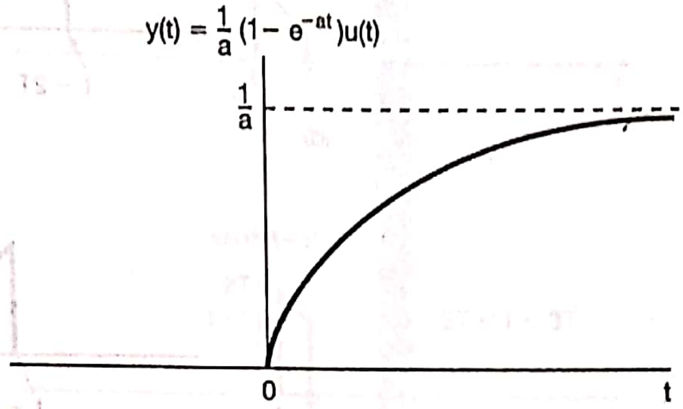


Figure 2.18 Response of the system in Example 2.6 with impulse response $h(t) = u(t)$ to the input $x(t) = e^{-at}u(t)$.

Example 2.7

Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

As in Example 2.4 for discrete-time convolution, it is convenient to consider the evaluation of $y(t)$ in separate intervals. In Figure 2.19, we have sketched $x(\tau)$ and have illustrated $h(t-\tau)$ in each of the intervals of interest. For $t < 0$ and for $t > 3T$, $x(\tau)h(t-\tau) = 0$ for all values of τ, and consequently, $y(t) = 0$. For the other intervals, the product $x(\tau)h(t-\tau)$ is as indicated in Figure 2.20. Thus, for these three intervals, the integration can be carried out graphically, with the result that

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases},$$

which is depicted in Figure 2.21.

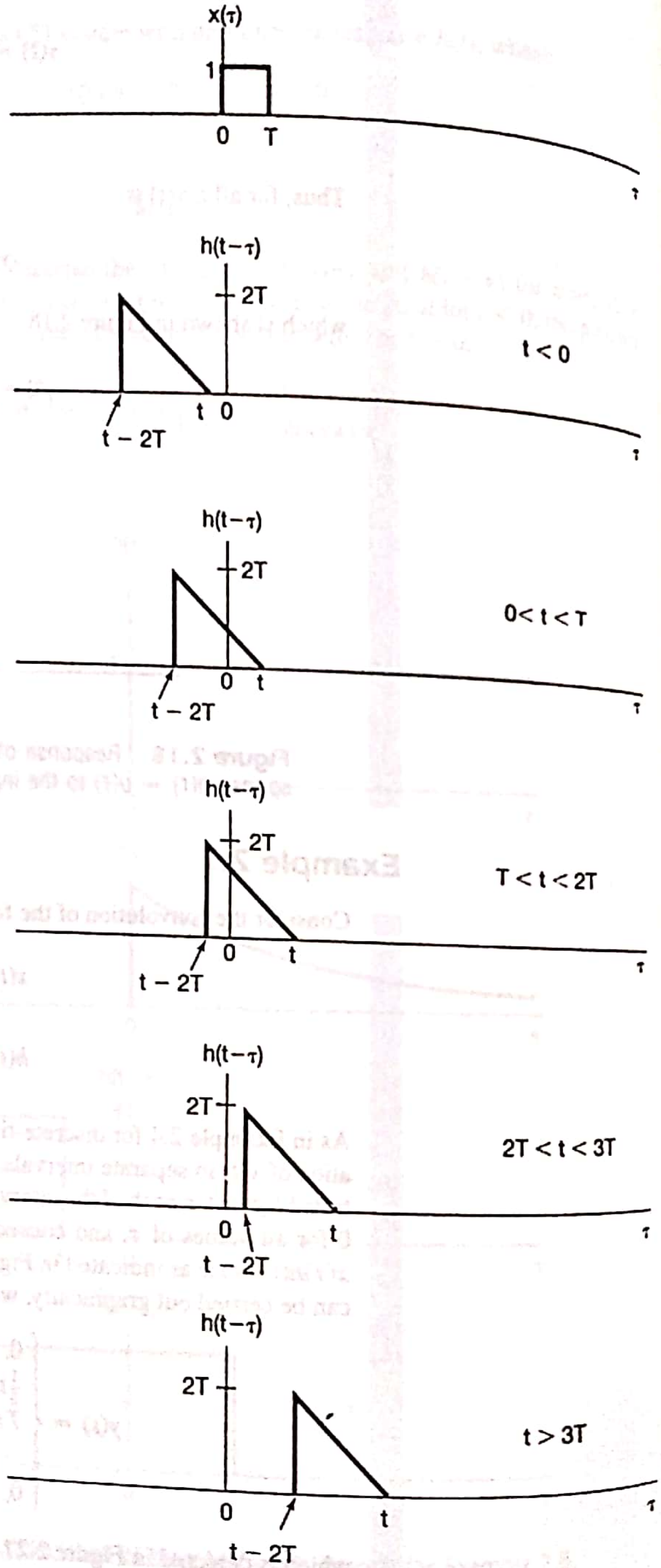


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

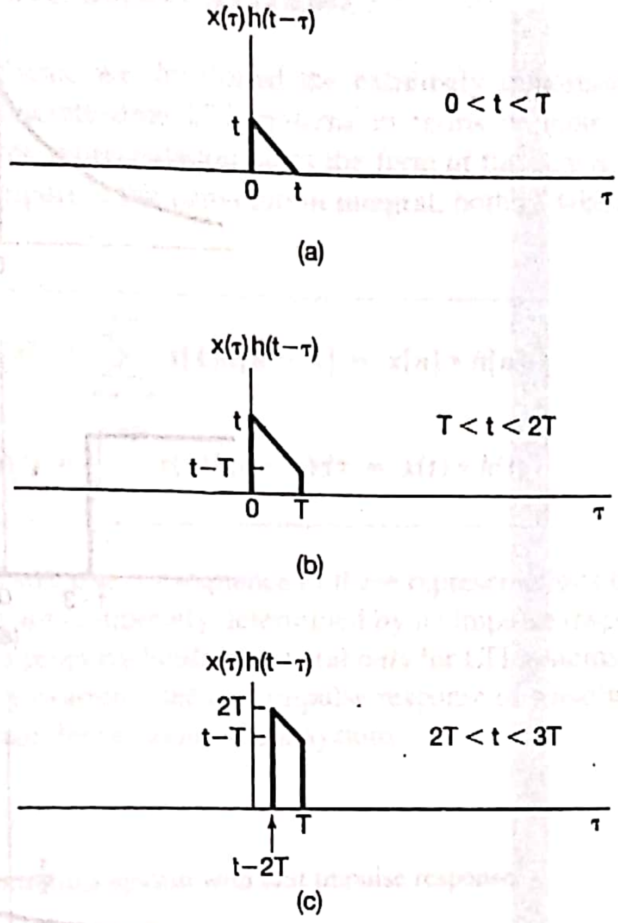


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

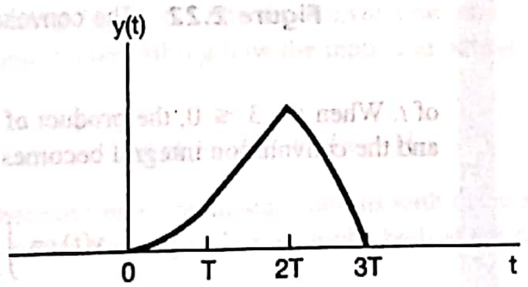


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

Example 2.8

Let $y(t)$ denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t), \tag{2.35}$$

$$h(t) = u(t-3). \tag{2.36}$$

The signals $x(\tau)$ and $h(t-\tau)$ are plotted as functions of τ in Figure 2.22(a). We first observe that these two signals have regions of nonzero overlap, regardless of the value

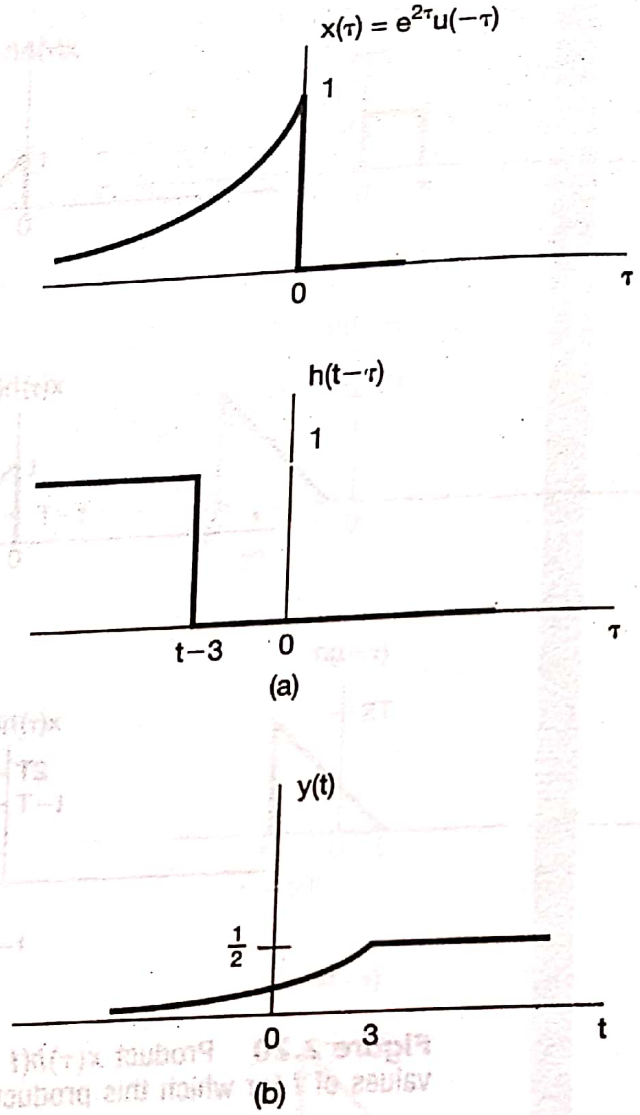


Figure 2.22 The convolution problem considered in Example 2.8.

of t. When $t-3 \leq 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < t-3$, and the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}. \tag{2.37}$$

For $t-3 \geq 0$, the product $x(\tau)h(t-\tau)$ is nonzero for $-\infty < \tau < 0$, so that the convolution integral is

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}. \tag{2.38}$$

The resulting signal $y(t)$ is plotted in Figure 2.22(b).

As these examples and those presented in Section 2.1 illustrate, the graphical interpretation of continuous-time and discrete-time convolution is of considerable value in visualizing the evaluation of convolution integrals and sums.

2.3 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

In the preceding two sections, we developed the extremely important representations of continuous-time and discrete-time LTI systems in terms of their unit impulse responses. In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral, both of which we repeat here for convenience:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n] \tag{2.39}$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \tag{2.40}$$

As we have pointed out, one consequence of these representations is that the characteristics of an LTI system are completely determined by its impulse response. It is important to emphasize that this property holds in general *only* for LTI systems. In particular, as illustrated in the following example, the unit impulse response of a nonlinear system does *not* completely characterize the behavior of the system.

Example 2.9

Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.41}$$

If the system is LTI, then eq. (2.41) completely determines its input-output behavior. In particular, by substituting eq. (2.41) into the convolution sum, eq. (2.39), we find the following explicit equation describing how the input and output of this LTI system are related:

$$y[n] = x[n] + x[n-1]. \tag{2.42}$$

On the other hand, there are *many* nonlinear systems with the same response—i.e., that given in eq. (2.41)—to the input $\delta[n]$. For example, both of the following systems have this property:

$$y[n] = (x[n] + x[n-1])^2,$$

$$y[n] = \max(x[n], x[n-1]).$$

Consequently, if the system is nonlinear it is not completely characterized by the impulse response in eq. (2.41).

The preceding example illustrates the fact that LTI systems have a number of properties not possessed by other systems, beginning with the very special representations that they have in terms of convolution sums and integrals. In the remainder of this section, we explore some of the most basic and important of these properties.

2.3.1 The Commutative Property

A basic property of convolution in both continuous and discrete time is that it is a *commutative* operation. That is, in discrete time

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k], \tag{2.43}$$

and in continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau. \tag{2.44}$$

These expressions can be verified in a straightforward manner by means of a substitution of variables in eqs. (2.39) and (2.40). For example, in the discrete-time case, if we let $r = n - k$ or, equivalently, $k = n - r$, eq. (2.39) becomes

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r] = h[n] * x[n]. \tag{2.45}$$

With this substitution of variables, the roles of $x[n]$ and $h[n]$ are interchanged. According to eq. (2.45), the output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$. For example, we could have calculated the convolution in Example 2.4 by first reflecting and shifting $x[k]$, then multiplying the signals $x[n-k]$ and $h[k]$, and finally summing the products for all values of k.

Similarly, eq. (2.44) can be verified by a change of variables, and the implications of this result in continuous time are the same: The output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI system with input $h(t)$ and unit impulse response $x(t)$. Thus, we could have calculated the convolution in Example 2.7 by reflecting and shifting $x(t)$, multiplying the signals $x(t-\tau)$ and $h(\tau)$, and integrating over $-\infty < \tau < +\infty$. In specific cases, one of the two forms for computing convolutions [i.e., eq. (2.39) or (2.43) in discrete time and eq. (2.40) or (2.44) in continuous time] may be easier to visualize, but both forms always result in the same answer.

2.3.2 The Distributive Property

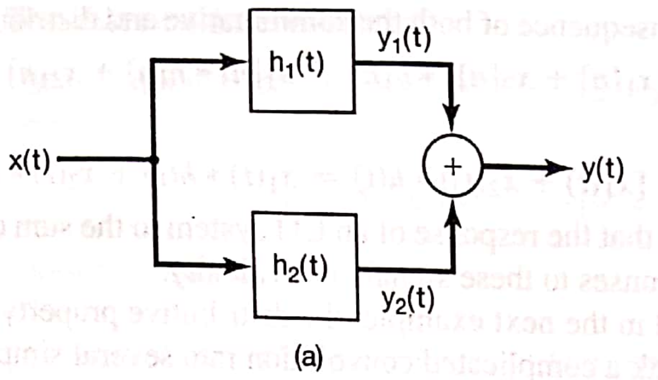
Another basic property of convolution is the *distributive* property. Specifically, convolution distributes over addition, so that in discrete time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n], \tag{2.46}$$

and in continuous time

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \tag{2.47}$$

This property can be verified in a straightforward manner.



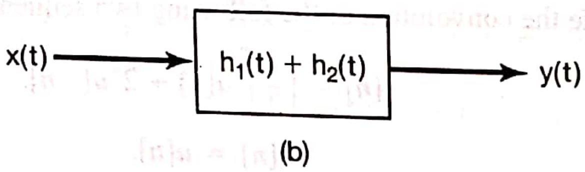


Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

The distributive property has a useful interpretation in terms of system interconnections. Consider two continuous-time LTI systems in parallel, as indicated in Figure 2.23(a). The systems shown in the block diagram are LTI systems with the indicated unit impulse responses. This pictorial representation is a particularly convenient way in which to denote LTI systems in block diagrams, and it also reemphasizes the fact that the impulse response of an LTI system completely characterizes its behavior.

The two systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs, and their outputs are added. Since

$$y_1(t) = x(t) * h_1(t)$$

and

$$y_2(t) = x(t) * h_2(t),$$

the system of Figure 2.23(a) has output

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t), \tag{2.48}$$

corresponding to the right-hand side of eq. (2.47). The system of Figure 2.23(b) has output

$$y(t) = x(t) * [h_1(t) + h_2(t)], \tag{2.49}$$

corresponding to the left-hand side of eq. (2.47). Applying eq. (2.47) to eq. (2.49) and comparing the result with eq. (2.48), we see that the systems in Figures 2.23(a) and (b) are identical.

There is an identical interpretation in discrete time, in which each of the signals in Figure 2.23 is replaced by a discrete-time counterpart (i.e., $x(t)$, $h_1(t)$, $h_2(t)$, $y_1(t)$, $y_2(t)$, and $y(t)$ are replaced by $x[n]$, $h_1[n]$, $h_2[n]$, $y_1[n]$, $y_2[n]$, and $y[n]$, respectively). In summary, then, by virtue of the distributive property of convolution, a parallel combination of LTI systems can be replaced by a single LTI system whose unit impulse response is the sum of the individual unit impulse responses in the parallel combination.

Also, as a consequence of both the commutative and distributive properties, we have

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \tag{2.50}$$

and

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t), \tag{2.51}$$

which simply state that the response of an LTI system to the sum of two inputs must equal the sum of the responses to these signals individually.

As illustrated in the next example, the distributive property of convolution can also be exploited to break a complicated convolution into several simpler ones.

Example 2.10

Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n], \tag{2.52}$$

$$h[n] = u[n]. \tag{2.53}$$

Note that the sequence $x[n]$ is nonzero along the entire time axis. Direct evaluation of such a convolution is somewhat tedious. Instead, we may use the distributive property to express $y[n]$ as the sum of the results of two simpler convolution problems. In particular, if we let $x_1[n] = (1/2)^n u[n]$ and $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]. \tag{2.54}$$

Using the distributive property of convolution, we may rewrite eq. (2.54) as

$$y[n] = y_1[n] + y_2[n], \tag{2.55}$$

where

$$y_1[n] = x_1[n] * h[n] \tag{2.56}$$

and

$$y_2[n] = x_2[n] * h[n]. \tag{2.57}$$

The convolution in eq. (2.56) for $y_1[n]$ can be obtained from Example 2.3 (with $\alpha = 1/2$), while $y_2[n]$ was evaluated in Example 2.5. Their sum is $y[n]$, which is shown in Figure 2.24.

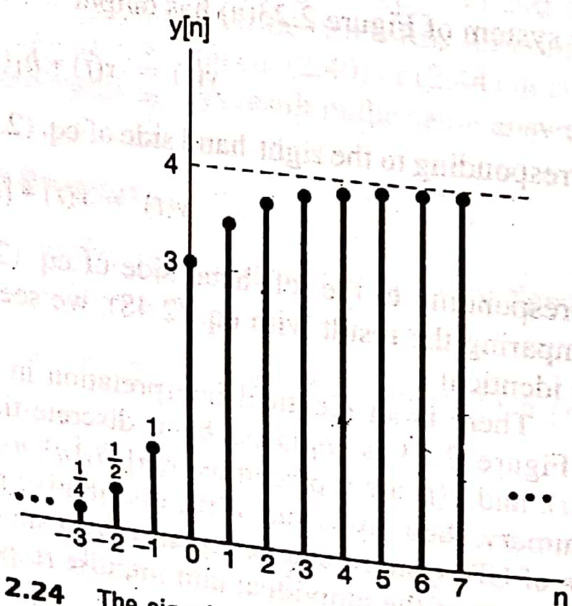


Figure 2.24 The signal $y[n] = x[n] * h[n]$ for Example 2.10.

2.3.3 The Associative Property

Another important and useful property of convolution is that it is *associative*. That is, in discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n], \tag{2.58}$$

and in continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t). \tag{2.59}$$

This property is proven by straightforward manipulations of the summations and integrals involved. Examples verifying it are given in Problem 2.43.

As a consequence of the associative property, the expressions

$$y[n] = x[n] * h_1[n] * h_2[n] \tag{2.60}$$

and

$$y(t) = x(t) * h_1(t) * h_2(t) \tag{2.61}$$

are unambiguous. That is, according to eqs. (2.58) and (2.59), it does not matter in which order we convolve these signals.

An interpretation of the associative property is illustrated for discrete-time systems in Figures 2.25(a) and (b). In Figure 2.25(a),

$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n]. \end{aligned}$$

In Figure 2.25(b),

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]). \end{aligned}$$

According to the associative property, the series interconnection of the two systems in Figure 2.25(a) is equivalent to the single system in Figure 2.25(b). This can be generalized to an arbitrary number of LTI systems in cascade, and the analogous interpretation and conclusion also hold in continuous time.

By using the commutative property together with the associative property, we find another very important property of LTI systems. Specifically, from Figures 2.25(a) and (b), we can conclude that the impulse response of the cascade of two LTI systems is the convolution of their individual impulse responses. Since convolution is commutative, we can compute this convolution of $h_1[n]$ and $h_2[n]$ in either order. Thus, Figures 2.25(b) and (c) are equivalent, and from the associative property, these are in turn equivalent to the system of Figure 2.25(d), which we note is a cascade combination of two systems as in Figure 2.25(a), but with the order of the cascade reversed. Consequently, the unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded. In fact, this holds for an arbitrary number of LTI systems in cascade: The order in which they are cascaded does not matter as far as the overall system impulse response is concerned. The same conclusions hold in continuous time as well.

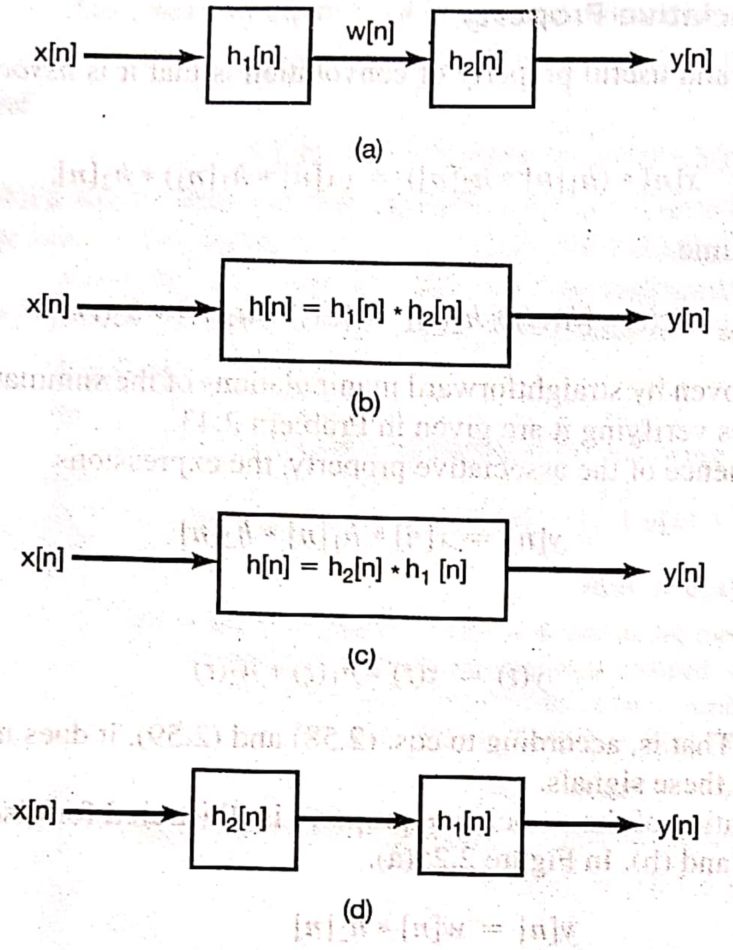


Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

It is important to emphasize that the behavior of LTI systems in cascade—and, in particular, the fact that the overall system response does not depend upon the order of the systems in the cascade—is very special to such systems. In contrast, the order in which nonlinear systems are cascaded cannot be changed, in general, without changing the overall response. For instance, if we have two memoryless systems, one being multiplication by 2 and the other squaring the input, then if we multiply first and square second, we obtain

$$y[n] = 4x^2[n].$$

However, if we multiply by 2 after squaring, we have

$$y[n] = 2x^2[n].$$

Thus, being able to interchange the order of systems in a cascade is a characteristic particular to LTI systems. In fact, as shown in Problem 2.51, we need both linearity *and* time invariance in order for this property to be true in general.

2.3.4 LTI Systems with and without Memory

As specified in Section 1.6.1, a system is memoryless if its output at any time depends only on the value of the input at that same time. From eq. (2.39), we see that the only way that this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$. In this case the impulse response has the form

$$h[n] = K\delta[n], \tag{2.62}$$

where $K = h[0]$ is a constant, and the convolution sum reduces to the relation

$$y[n] = Kx[n]. \tag{2.63}$$

If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory. An example of an LTI system with memory is the system given by eq. (2.42). The impulse response for this system, given in eq. (2.41), is nonzero for $n = 1$.

From eq. (2.40), we can deduce similar properties for continuous-time LTI systems with and without memory. In particular, a continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$, and such a memoryless LTI system has the form

$$y(t) = Kx(t) \tag{2.64}$$

for some constant K and has the impulse response

$$h(t) = K\delta(t). \tag{2.65}$$

Note that if $K = 1$ in eqs. (2.62) and (2.65), then these systems become identity systems, with output equal to the input and with unit impulse response equal to the unit impulse. In this case, the convolution sum and integral formulas imply that

$$x[n] = x[n] * \delta[n]$$

and

$$x(t) = x(t) * \delta(t),$$

which reduce to the sifting properties of the discrete-time and continuous-time unit impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

2.3.5 Invertibility of LTI Systems

Consider a continuous-time LTI system with impulse response $h(t)$. Based on the discussion in Section 1.6.2, this system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system. Furthermore, if an LTI system is invertible, then it has an LTI inverse. (See Problem 2.50.) Therefore, we have the picture shown in Figure 2.26. We are given a system with impulse response $h(t)$. The inverse system, with impulse response $h_1(t)$, results in $w(t) = x(t)$—such that the series interconnection in Figure 2.26(a) is identical to the

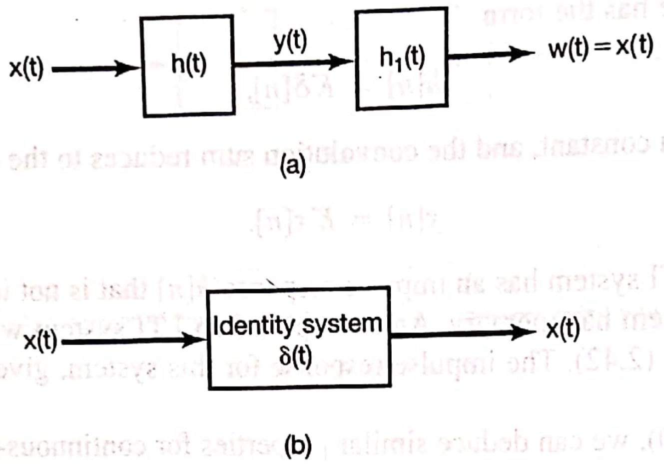


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

identity system in Figure 2.26(b). Since the overall impulse response in Figure 2.26(a) is $h(t) * h_1(t)$, we have the condition that $h_1(t)$ must satisfy for it to be the impulse response of the inverse system, namely,

$$h(t) * h_1(t) = \delta(t). \tag{2.66}$$

Similarly, in discrete time, the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]. \tag{2.67}$$

The following two examples illustrate invertibility and the construction of an inverse system.

Example 2.11

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0). \tag{2.68}$$

Such a system is a *delay* if $t_0 > 0$ and an *advance* if $t_0 < 0$. For example, if $t_0 > 0$, then the output at time t equals the value of the input at the earlier time $t - t_0$. If $t_0 = 0$, the system in eq. (2.68) is the identity system and thus is memoryless. For any other value of t_0, this system has memory, as it responds to the value of the input at a time other than the current time.

The impulse response for the system can be obtained from eq. (2.68) by taking the input equal to $\delta(t)$, i.e.,

$$h(t) = \delta(t - t_0). \tag{2.69}$$

Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0). \tag{2.70}$$

That is, the convolution of a signal with a shifted impulse simply shifts the signal.

To recover the input from the output, i.e., to invert the system, all that is required is to shift the output back. The system with this compensating time shift is then the inverse system. That is, if we take

$$h_1(t) = \delta(t + t_0),$$

then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

Similarly, a pure time shift in discrete time has the unit impulse response $\delta[n-n_0]$, so that convolving a signal with a shifted impulse is the same as shifting the signal. Furthermore, the inverse of the LTI system with impulse response $\delta[n - n_0]$ is the LTI system that shifts the signal in the opposite direction by the same amount—i.e., the LTI system with impulse response $\delta[n + n_0]$.

Example 2.12

Consider an LTI system with impulse response

$$h[n] = u[n]. \tag{2.71}$$

Using the convolution sum, we can calculate the response of this system to an arbitrary input:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]u[n-k]. \tag{2.72}$$

Since $u[n - k]$ is 0 for $n - k < 0$ and 1 for $n - k \geq 0$, eq. (2.72) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{2.73}$$

That is, this system, which we first encountered in Section 1.6.1 [see eq. (1.92)], is a summer or accumulator that computes the running sum of all the values of the input up to the present time. As we saw in Section 1.6.2, such a system is invertible, and its inverse, as given by eq. (1.99), is

$$y[n] = x[n] - x[n-1], \tag{2.74}$$

which is simply a *first difference* operation. Choosing $x[n] = \delta[n]$, we find that the impulse response of the inverse system is

$$h_1[n] = \delta[n] - \delta[n-1]. \tag{2.75}$$

As a check that $h[n]$ in eq. (2.71) and $h_1[n]$ in eq. (2.75) are indeed the impulse responses of LTI systems that are inverses of each other, we can verify eq. (2.67) by direct calculation:

$$\begin{aligned} h[n] * h_1[n] &= u[n] * \{\delta[n] - \delta[n-1]\} \\ &= u[n] * \delta[n] - u[n] * \delta[n-1] \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned} \tag{2.76}$$

2.3.6 Causality for LTI Systems

In Section 1.6.3, we introduced the property of causality: The output of a causal system depends only on the present and past values of the input to the system. By using the convolution sum and integral, we can relate this property to a corresponding property of the impulse response of an LTI system. Specifically, in order for a discrete-time LTI system to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$. From eq. (2.39), we see that for this to be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero. This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0. \tag{2.77}$$

According to eq. (2.77), the impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality. More generally, as shown in Problem 1.44, causality for a linear system is equivalent to the condition of *initial rest*; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time. It is important to emphasize that the equivalence of causality and the condition of initial rest applies only to linear systems. For example, as discussed in Section 1.6.6, the system $y[n] = 2x[n] + 3$ is not linear. However, it is causal and, in fact, memoryless. On the other hand, if $x[n] = 0$, $y[n] = 3 \neq 0$, so it does not satisfy the condition of initial rest.

For a causal discrete-time LTI system, the condition in eq. (2.77) implies that the convolution sum representation in eq. (2.39) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k], \tag{2.78}$$

and the alternative equivalent form, eq. (2.43), becomes

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]. \tag{2.79}$$

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0, \tag{2.80}$$

and in this case the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{2.81}$$

Both the accumulator ($h[n] = u[n]$) and its inverse ($h[n] = \delta[n] - \delta[n-1]$), described in Example 2.12, satisfy eq. (2.77) and therefore are causal. The pure time shift with impulse response $h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$ (when the time shift is a delay), but is noncausal for $t_0 < 0$ (in which case the time shift is an advance, so that the output anticipates future values of the input).

Finally, while causality is a property of systems, it is common terminology to refer to a signal as being causal if it is zero for $n < 0$ or $t < 0$. The motivation for this terminology comes from eqs. (2.77) and (2.80): Causality of an LTI system is equivalent to its impulse response being a causal signal.

2.3.7 Stability for LTI Systems

Recall from Section 1.6.4 that a system is *stable* if every bounded input produces a bounded output. In order to determine conditions under which LTI systems are stable, consider an input $x[n]$ that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n. \tag{2.82}$$

Suppose that we apply this input to an LTI system with unit impulse response $h[n]$. Then, using the convolution sum, we obtain an expression for the magnitude of the output:

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|. \tag{2.83}$$

Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, it follows from eq. (2.83) that

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|. \tag{2.84}$$

From eq. (2.82), $|x[n-k]| < B$ for all values of k and n. Together with eq. (2.84), this implies that

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n. \tag{2.85}$$

From eq. (2.85), we can conclude that if the impulse response is *absolutely summable*, that is, if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty, \tag{2.86}$$

then $y[n]$ is bounded in magnitude, and hence, the system is stable. Therefore, eq. (2.86) is a sufficient condition to guarantee the stability of a discrete-time LTI system. In fact, this condition is also a necessary condition, since, as shown in Problem 2.49, if eq. (2.86) is not satisfied, there are bounded inputs that result in unbounded outputs. Thus, the stability of a discrete-time LTI system is completely equivalent to eq. (2.86).

In continuous time, we obtain an analogous characterization of stability in terms of the impulse response of an LTI system. Specifically, if $|x(t)| < B$ for all t, then, in analogy with eqs. (2.83)–(2.85), it follows that

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right| \\ &\le \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau \\ &\le B \int_{-\infty}^{+\infty} |h(\tau)|d\tau. \end{aligned}$$

Therefore, the system is stable if the impulse response is *absolutely integrable*, i.e., if

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty. \tag{2.87}$$

As in discrete time, if eq. (2.87) is not satisfied, there are bounded inputs that produce unbounded outputs; therefore, the stability of a continuous-time LTI system is equivalent to eq. (2.87). The use of eqs (2.86) and (2.87) to test for stability is illustrated in the next two examples.

Example 2.13

Consider a system that is a pure time shift in either continuous time or discrete time. Then, in discrete time

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n-n_0]| = 1, \tag{2.88}$$

while in continuous time

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \int_{-\infty}^{+\infty} |\delta(\tau-t_0)|d\tau = 1, \tag{2.89}$$

and we conclude that both of these systems are stable. This should not be surprising, since if a signal is bounded in magnitude, so is any time-shifted version of that signal.

Now consider the accumulator described in Example 2.12. As we discussed in Section 1.6.4, this is an unstable system, since, if we apply a constant input to an accumulator, the output grows without bound. That this system is unstable can also be seen from the fact that its impulse response $u[n]$ is not absolutely summable:

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} u[n] = \infty.$$

Similarly, consider the integrator, the continuous-time counterpart of the accumulator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau. \tag{2.90}$$

This is an unstable system for precisely the same reason as that given for the accumulator; i.e., a constant input gives rise to an output that grows without bound. The impulse response for the integrator can be found by letting $x(t) = \delta(t)$, in which case

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

and

$$\int_{-\infty}^{+\infty} |u(\tau)|d\tau = \int_{0}^{+\infty} d\tau = \infty.$$

Since the impulse response is not absolutely integrable, the system is not stable.

2.3.8 The Unit Step Response of an LTI System

Up to now, we have seen that the representation of an LTI system in terms of its unit impulse response allows us to obtain very explicit characterizations of system properties. Specifically, since $h[n]$ or $h(t)$ completely determines the behavior of an LTI system, we have been able to relate system properties such as stability and causality to properties of the impulse response.

There is another signal that is also used quite often in describing the behavior of LTI systems: the *unit step response*, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$. We will find it useful on occasion to refer to the step response, and therefore, it is worthwhile relating it to the impulse response. From the convolution-sum representation, the step response of a discrete-time LTI system is the convolution of the unit step with the impulse response; that is,

$$s[n] = u[n] * h[n].$$

However, by the commutative property of convolution, $s[n] = h[n] * u[n]$, and therefore, $s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$. As we have seen in Example 2.12, $u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{2.91}$$

From this equation and from Example 2.12, it is clear that $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1]. \tag{2.92}$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response [eq. (2.91)]. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response [eq. (2.92)].

Similarly, in continuous time, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \tag{2.93}$$

and from eq. (2.93), the unit impulse response is the first derivative of the unit step response,[1] or

$$h(t) = \frac{ds(t)}{dt} = s'(t). \tag{2.94}$$

Therefore, in both continuous and discrete time, the unit step response can also be used to characterize an LTI system, since we can calculate the unit impulse response from it. In Problem 2.45, expressions analogous to the convolution sum and convolution integral are derived for the representations of an LTI system in terms of its unit step response.

2.4 CAUSAL LTI SYSTEMS DESCRIBED BY DIFFERENTIAL AND DIFFERENCE EQUATIONS

An extremely important class of continuous-time systems is that for which the input and output are related through a *linear constant-coefficient differential equation*. Equations of this type arise in the description of a wide variety of systems and physical phenomena. For example, as we illustrated in Chapter 1, the response of the *RC* circuit in Figure 1.1 and the motion of a vehicle subject to acceleration inputs and frictional forces, as depicted in Figure 1.2, can both be described through linear constant-coefficient differential equations. Similar differential equations arise in the description of mechanical systems containing restoring and damping forces, in the kinetics of chemical reactions, and in many other contexts as well.

Correspondingly, an important class of discrete-time systems is that for which the input and output are related through a *linear constant-coefficient difference equation*. Equations of this type are used to describe the sequential behavior of many different processes. For instance, in Example 1.10 we saw how difference equations arise in describing the accumulation of savings in a bank account, and in Example 1.11 we saw how they can be used to describe a digital simulation of a continuous-time system described by a differential equation. Difference equations also arise quite frequently in the specification of discrete-time systems designed to perform particular operations on the input signal. For example, the system that calculates the difference between successive input values, as in eq. (1.99), and the system described by eq. (1.104) that computes the average value of the input over an interval are described by difference equations.

Throughout this book, there will be many occasions in which we will consider and examine systems described by linear constant-coefficient differential and difference equations. In this section we take a first look at these systems to introduce some of the basic ideas involved in solving differential and difference equations and to uncover and explore some of the properties of systems described by such equations. In subsequent chapters, we develop additional tools for the analysis of signals and systems that will add considerably both to our ability to analyze systems described by such equations and to our understanding of their characteristics and behavior.

[1] Throughout this book, we will use both the notations indicated in eq. (2.94) to denote first derivatives. Analogous notation will also be used for higher derivatives.

2.4.1 Linear Constant-Coefficient Differential Equations

To introduce some of the important ideas concerning systems specified by linear constant-coefficient differential equations, let us consider a first-order differential equation as in eq. (1.85), viz.,

$$\frac{dy(t)}{dt} + 2y(t) = x(t), \tag{2.95}$$

where $y(t)$ denotes the output of the system and $x(t)$ is the input. For example, comparing eq. (2.95) to the differential equation (1.84) for the velocity of a vehicle subject to applied and frictional forces, we see that eq. (2.95) would correspond exactly to this system if $y(t)$ were identified with the vehicle's velocity $v(t)$, if $x(t)$ were taken as the applied force $f(t)$, and if the parameters in eq. (1.84) were normalized in units such that $b/m = 2$ and $1/m = 1$.

A very important point about differential equations such as eq. (2.95) is that they provide an *implicit* specification of the system. That is, they describe a relationship between the input and the output, rather than an explicit expression for the system output as a function of the input. In order to obtain an explicit expression, we must solve the differential equation. To find a solution, we need more information than that provided by the differential equation alone. For example, to determine the speed of an automobile at the end of a 10-second interval when it has been subjected to a constant acceleration of 1 m/sec^2 for 10 seconds, we would also need to know how fast the vehicle was moving at the *start* of the interval. Similarly, if we are told that a constant source voltage of 1 volt is applied to the *RC* circuit in Figure 1.1 for 10 seconds, we cannot determine what the capacitor voltage is at the end of that interval without also knowing what the initial capacitor voltage is.

More generally, to solve a differential equation, we must specify one or more auxiliary conditions, and once these are specified, we can then, in principle, obtain an explicit expression for the output in terms of the input. In other words, a differential equation such as eq. (2.95) describes a constraint between the input and the output of a system, but to characterize the system completely, we must also specify auxiliary conditions. Different choices for these auxiliary conditions then lead to different relationships between the input and the output. For the most part, in this book we will focus on the use of differential equations to describe causal LTI systems, and for such systems the auxiliary conditions take a particular, simple form. To illustrate this and to uncover some of the basic properties of the solutions to differential equations, let us take a look at the solution of eq. (2.95) for a specific input signal $x(t)$.[2]

[2]Our discussion of the solution of linear constant-coefficient differential equations is brief, since we assume that the reader has some familiarity with this material. For review, we recommend a text on the solution of ordinary differential equations, such as *Ordinary Differential Equations* (3rd ed.), by G. Birkhoff and G.-C. Rota (New York: John Wiley and Sons, 1978), or *Elementary Differential Equations* (3rd ed.), by W.E. Boyce and R.C. DiPrima (New York: John Wiley and Sons, 1977). There are also numerous texts that discuss differential equations in the context of circuit theory. See, for example, *Basic Circuit Theory*, by L.O. Chua, C.A. Desoer, and E.S. Kuh (New York: McGraw-Hill Book Company, 1987). As mentioned in the text, in the following chapters we present other very useful methods for solving linear differential equations that will be sufficient for our purposes. In addition, a number of exercises involving the solution of differential equations are included in the problems at the end of the chapter.

Example 2.14

Consider the solution of eq. (2.95) when the input signal is

$$x(t) = Ke^{3t}u(t), \tag{2.96}$$

where K is a real number.

The complete solution to eq. (2.96) consists of the sum of a *particular solution*, $y_p(t)$, and a *homogeneous solution*, $y_h(t)$, i.e.,

$$y(t) = y_p(t) + y_h(t), \tag{2.97}$$

where the particular solution satisfies eq. (2.95) and $y_h(t)$ is a solution of the homogeneous differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0. \tag{2.98}$$

A common method for finding the particular solution for an exponential input signal as in eq. (2.96) is to look for a so-called *forced response*—i.e., a signal of the same form as the input. With regard to eq. (2.95), since $x(t) = Ke^{3t}$ for $t > 0$, we hypothesize a solution for $t > 0$ of the form

$$y_p(t) = Ye^{3t}, \tag{2.99}$$

where Y is a number that we must determine. Substituting eqs. (2.96) and (2.99) into eq. (2.95) for $t > 0$ yields

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}. \tag{2.100}$$

Canceling the factor e^{3t} from both sides of eq. (2.100), we obtain

$$3Y + 2Y = K, \tag{2.101}$$

or

$$Y = \frac{K}{5}, \tag{2.102}$$

so that

$$y_p(t) = \frac{K}{5}e^{3t}, \quad t > 0. \tag{2.103}$$

In order to determine $y_h(t)$, we hypothesize a solution of the form

$$y_h(t) = Ae^{st}. \tag{2.104}$$

Substituting this into eq. (2.98) gives

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s + 2) = 0. \tag{2.105}$$

From this equation, we see that we must take $s = -2$ and that Ae^{-2t} is a solution to eq. (2.98) for *any* choice of A. Utilizing this fact and eq. (2.103) in eq. (2.97), we find that the solution of the differential equation for $t > 0$ is

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \quad t > 0. \tag{2.106}$$

As noted earlier, the differential equation (2.95) by itself does not specify uniquely the response $y(t)$ to the input $x(t)$ in eq. (2.96). In particular, the constant A in eq. (2.106) has not yet been determined. In order for the value of A to be determined, we need to specify an auxiliary condition in addition to the differential equation (2.95). As explored in Problem 2.34, different choices for this auxiliary condition lead to different solutions $y(t)$ and, consequently, to different relationships between the input and the output. As we have indicated, for the most part in this book we focus on differential and difference equations used to describe systems that are LTI and causal, and in this case the auxiliary condition takes the form of the condition of initial rest. That is, as shown in Problem 1.44, for a causal LTI system, if $x(t) = 0$ for $t < t_0$, then $y(t)$ must also equal 0 for $t < t_0$. From eq. (2.96), we see that for our example $x(t) = 0$ for $t < 0$, and thus, the condition of initial rest implies that $y(t) = 0$ for $t < 0$. Evaluating eq. (2.106) at $t = 0$ and setting $y(0) = 0$ yields

$$0 = A + \frac{K}{5},$$

or

$$A = -\frac{K}{5}.$$

Thus, for $t > 0$,

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right], \tag{2.107}$$

while for $t < 0$, $y(t) = 0$, because of the condition of initial rest. Combining these two cases, we obtain the full solution

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right]u(t). \tag{2.108}$$

Example 2.14 illustrates several very important points concerning linear constant-coefficient differential equations and the systems they represent. First, the response to an input $x(t)$ will generally consist of the sum of a particular solution to the differential equation and a homogeneous solution—i.e., a solution to the differential equation with the input set to zero. The homogeneous solution is often referred to as the *natural response* of the system. The natural responses of simple electrical circuits and mechanical systems are explored in Problems 2.61 and 2.62.

In Example 2.14 we also saw that, in order to determine completely the relationship between the input and the output of a system described by a differential equation such as eq. (2.95), we must specify auxiliary conditions. An implication of this fact, which is illustrated in Problem 2.34, is that different choices of auxiliary conditions lead to different relationships between the input and the output. As we illustrated in the example, for the most part we will use the condition of initial rest for systems described by differential equations. In the example, since the input was 0 for $t < 0$, the condition of initial rest implied the initial condition $y(0) = 0$. As we have stated, and as illustrated in

Problem 2.33, under the condition of initial rest the system described by eq. (2.95) is LTI and causal.[3] For example, if we multiply the input in eq. (2.96) by 2, the resulting output would be twice the output in eq. (2.108).

It is important to emphasize that the condition of initial rest does not specify a zero initial condition at a fixed point in time, but rather adjusts this point in time so that the response is zero *until* the input becomes nonzero. Thus, if $x(t) = 0$ for $t \le t_0$ for the causal LTI system described by eq. (2.95), then $y(t) = 0$ for $t \le t_0$, and we would use the initial condition $y(t_0) = 0$ to solve for the output for $t > t_0$. As a physical example, consider again the circuit in Figure 1.1, also discussed in Example 1.8. Initial rest for this example corresponds to the statement that, until we connect a nonzero voltage source to the circuit, the capacitor voltage is zero. Thus, if we begin to use the circuit at noon today, the initial capacitor voltage as we connect the voltage source at noon today is zero. Similarly, if we begin to use the circuit at noon tomorrow instead, the initial capacitor voltage as we connect the voltage source at noon tomorrow is zero.

This example also provides us with some intuition as to why the condition of initial rest makes a system described by a linear constant-coefficient differential equation time invariant. For example, if we perform an experiment on the circuit, starting from initial rest, then, assuming that the coefficients R and C don't change over time, we would expect to get the same results whether we ran the experiment today or tomorrow. That is, if we perform identical experiments on the two days, where the circuit starts from initial rest at noon on each day, then we would expect to see identical responses—i.e., responses that are simply time-shifted by one day with respect to each other.

While we have used the first-order differential equation (2.95) as the vehicle for the discussion of these issues, the same ideas extend directly to systems described by higher order differential equations. A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{2.109}$$

The order refers to the highest derivative of the output $y(t)$ appearing in the equation. In the case when $N = 0$, eq. (2.109) reduces to

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{2.110}$$

In this case, $y(t)$ is an explicit function of the input $x(t)$ and its derivatives. For $N \ge 1$, eq. (2.109) specifies the output implicitly in terms of the input. In this case, the analysis of the equation proceeds just as in our discussion of the first-order differential equation in Example 2.14. The solution $y(t)$ consists of two parts—a particular solution to eq. (2.109)

[3] In fact, as is also shown in Problem 2.34, if the initial condition for eq. (2.95) is nonzero, the resulting system is incrementally linear. That is, the overall response can be viewed, much as in Figure 1.48, as the superposition of the response to the initial conditions alone (with input set to 0) and the response to the input with an initial condition of 0 (i.e., the response of the causal LTI system described by eq. (2.95)).

plus a solution to the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \tag{2.111}$$

The solutions to this equation are referred to as the *natural responses* of the system.

As in the first-order case, the differential equation (2.109) does not completely specify the output in terms of the input, and we need to identify auxiliary conditions to determine completely the input-output relationship for the system. Once again, different choices for these auxiliary conditions result in different input-output relationships, but for the most part, in this book we will use the condition of initial rest when dealing with systems described by differential equations. That is, if $x(t) = 0$ for $t \leq t_0$, we assume that $y(t) = 0$ for $t \leq t_0$, and therefore, the response for $t > t_0$ can be calculated from the differential equation (2.109) with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0. \tag{2.112}$$

Under the condition of initial rest, the system described by eq. (2.109) is causal and LTI. Given the initial conditions in eq. (2.112), the output $y(t)$ can, in principle, be determined by solving the differential equation in the manner used in Example 2.14 and further illustrated in several problems at the end of the chapter. However, in Chapters 4 and 9 we will develop some tools for the analysis of continuous-time LTI systems that greatly facilitate the solution of differential equations and, in particular, provide us with powerful methods for analyzing and characterizing the properties of systems described by such equations.

2.4.2 Linear Constant-Coefficient Difference Equations

The discrete-time counterpart of eq. (2.109) is the Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{2.113}$$

An equation of this type can be solved in a manner exactly analogous to that for differential equations. (See Problem 2.32.)[4] Specifically, the solution $y[n]$ can be written as the sum of a particular solution to eq. (2.113) and a solution to the homogeneous equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0. \tag{2.114}$$

[4]For a detailed treatment of the methods for solving linear constant-coefficient difference equations, see *Finite Difference Equations*, by H. Levy and F. Lessman (New York: Macmillan, Inc., 1961), or *Finite Difference Equations and Simulations* (Englewood Cliffs, NJ: Prentice-Hall, 1968) by F. B. Hildebrand. In Chapter 6, we present another method for solving difference equations that greatly facilitates the analysis of linear time-invariant systems that are so described. In addition, we refer the reader to the problems at the end of this chapter that deal with the solution of difference equations.

The solutions to this homogeneous equation are often referred to as the natural responses of the system described by eq. (2.113).

As in the continuous-time case, eq. (2.113) does not completely specify the output in terms of the input. To do this, we must also specify some auxiliary conditions. While there are many possible choices for auxiliary conditions, leading to different input-output relationships, we will focus for the most part on the condition of initial rest—i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$ as well. With initial rest, the system described by eq. (2.113) is LTI and causal.

Although all of these properties can be developed following an approach that directly parallels our discussion for differential equations, the discrete-time case offers an alternative path. This stems from the observation that eq. (2.113) can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}. \tag{2.115}$$

Equation (2.115) directly expresses the output at time n in terms of previous values of the input and output. From this, we can immediately see the need for auxiliary conditions. In order to calculate $y[n]$, we need to know $y[n-1], \ldots, y[n-N]$. Therefore, if we are given the input for all n and a set of auxiliary conditions such as $y[-N], y[-N+1], \ldots, y[-1]$, eq. (2.115) can be solved for successive values of $y[n]$.

An equation of the form of eq. (2.113) or eq. (2.115) is called a *recursive equation*, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case when $N = 0$, eq. (2.115) reduces to

$$y[n] = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]. \tag{2.116}$$

This is the discrete-time counterpart of the continuous-time system given in eq. (2.110). Here, $y[n]$ is an explicit function of the present and previous values of the input. For this reason, eq. (2.116) is often called a *nonrecursive equation*, since we do not recursively use previously computed values of the output to compute the present value of the output. Therefore, just as in the case of the system given in eq. (2.110), we do not need auxiliary conditions in order to determine $y[n]$. Furthermore, eq. (2.116) describes an LTI system, and by direct computation, the impulse response of this system is found to be

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \tag{2.117}$$

That is, eq. (2.116) is nothing more than the convolution sum. Note that the impulse response for it has finite duration; that is, it is nonzero only over a finite time interval. Because of this property, the system specified by eq. (2.116) is often called a *finite impulse response (FIR) system*.

Although we do not require auxiliary conditions for the case of $N = 0$, such conditions are needed for the recursive case when $N \ge 1$. To illustrate the solution of such an equation, and to gain some insight into the behavior and properties of recursive difference equations, let us examine the following simple example:

Example 2.15

Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]. \tag{2.118}$$

Eq. (2.118) can also be expressed in the form

$$y[n] = x[n] + \frac{1}{2}y[n-1], \tag{2.119}$$

highlighting the fact that we need the previous value of the output, $y[n-1]$, to calculate the current value. Thus, to begin the recursion, we need an initial condition.

For example, suppose that we impose the condition of initial rest and consider the input

$$x[n] = K\delta[n]. \tag{2.120}$$

In this case, since $x[n] = 0$ for $n \leq -1$, the condition of initial rest implies that $y[n] = 0$ for $n \leq -1$, so that we have as an initial condition $y[-1] = 0$. Starting from this initial condition, we can solve for successive values of $y[n]$ for $n \geq 0$ as follows:

$$y[0] = x[0] + \frac{1}{2}y[-1] = K, \tag{2.121}$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K, \tag{2.122}$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K, \tag{2.123}$$

$$\vdots$$

$$y[n] = x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K. \tag{2.124}$$

Since the system specified by eq. (2.118) and the condition of initial rest is LTI, its input-output behavior is completely characterized by its impulse response. Setting $K = 1$, we see that the impulse response for the system considered in this example is

$$h[n] = \left(\frac{1}{2}\right)^n u[n]. \tag{2.125}$$

Note that the causal LTI system in Example 2.15 has an impulse response of infinite duration. In fact, if $N \geq 1$ in eq. (2.113), so that the difference equation is recursive, it is usually the case that the LTI system corresponding to this equation together with the condition of initial rest will have an impulse response of infinite duration. Such systems are commonly referred to as *infinite impulse response (IIR) systems.*

As we have indicated, for the most part we will use recursive difference equations in the context of describing and analyzing systems that are linear, time-invariant, and causal, and consequently, we will usually make the assumption of initial rest. In Chapters 5 and 10 we will develop tools for the analysis of discrete-time systems that will provide us with very useful and efficient methods for solving linear constant-coefficient difference equations and for analyzing the properties of the systems that they describe.

2.4.3 Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

An important property of systems described by linear constant-coefficient difference and differential equations is that they can be represented in very simple and natural ways in terms of block diagram interconnections of elementary operations. This is significant for a number of reasons. One is that it provides a pictorial representation which can add to our understanding of the behavior and properties of these systems. In addition, such representations can be of considerable value for the simulation or implementation of the systems. For example, the block diagram representation to be introduced in this section for continuous-time systems is the basis for early analog computer simulations of systems described by differential equations, and it can also be directly translated into a program for the simulation of such a system on a digital computer. In addition, the corresponding representation for discrete-time difference equations suggests simple and efficient ways in which the systems that the equations describe can be implemented in digital hardware. In this section, we illustrate the basic ideas behind these block diagram representations by constructing them for the causal first-order systems introduced in Examples 1.8–1.11. In Problems 2.57–2.60 and Chapters 9 and 10, we consider block diagrams for systems described by other, more complex differential and difference equations.

We begin with the discrete-time case and, in particular, the causal system described by the first-order difference equation

$$y[n] + ay[n-1] = bx[n]. \tag{2.126}$$

To develop a block diagram representation of this system, note that the evaluation of eq. (2.126) requires three basic operations: addition, multiplication by a coefficient, and delay (to capture the relationship between $y[n]$ and $y[n-1]$). Thus, let us define three basic network elements, as indicated in Figure 2.27. To see how these basic elements can be used to represent the causal system described by eq. (2.126), we rewrite this equation in the form that directly suggests a recursive algorithm for computing successive values of the output $y[n]$:

$$y[n] = -ay[n-1] + bx[n]. \tag{2.127}$$

This algorithm is represented pictorially in Figure 2.28, which is an example of a feedback system, since the output is fed back through a delay and a multiplication by a coefficient and is then added to $bx[n]$. The presence of feedback is a direct consequence of the recursive nature of eq. (2.127).

The block diagram in Figure 2.28 makes clear the required memory in this system and the consequent need for initial conditions. In particular, a delay corresponds to a memory element, as the element must retain the previous value of its input. Thus, the initial value of this memory element serves as a necessary initial condition for the recursive calculation specified pictorially in Figure 2.28 and mathematically in eq. (2.127). Of course, if the system described by eq. (2.126) is initially at rest, the initial value stored in the memory element is zero.

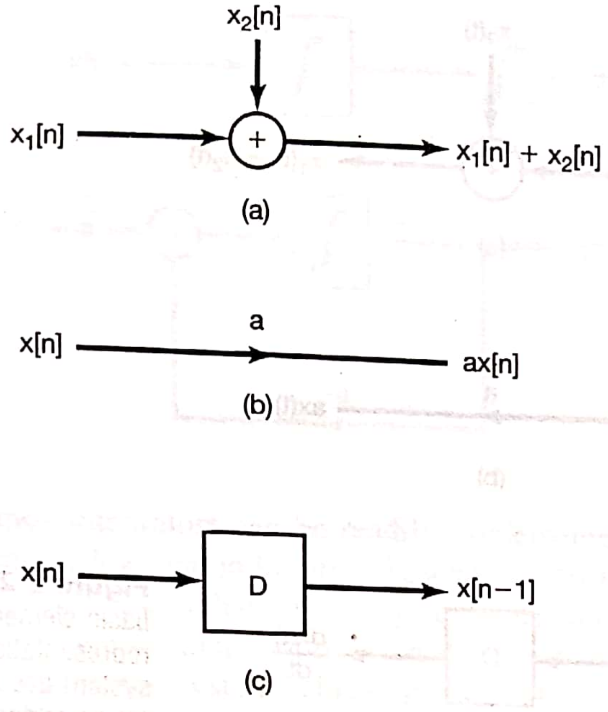


Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

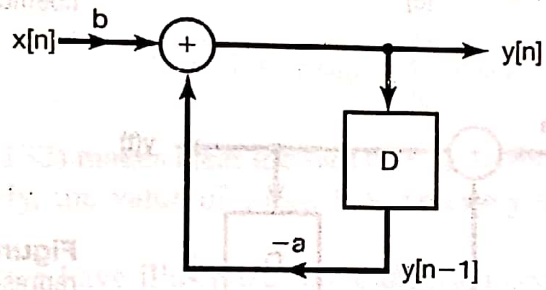


Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t). \tag{2.128}$$

As a first attempt at defining a block diagram representation for this system, let us rewrite it as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t). \tag{2.129}$$

The right-hand side of this equation involves three basic operations: addition, multiplication by a coefficient, and differentiation. Therefore, if we define the three basic network elements indicated in Figure 2.29, we can consider representing eq. (2.129) as an interconnection of these basic elements in a manner analogous to that used for the discrete-time system described previously, resulting in the block diagram of Figure 2.30.

While the latter figure is a valid representation of the causal system described by eq. (2.128), it is not the representation that is most frequently used or the representation that leads directly to practical implementations, since differentiators are both difficult to implement and extremely sensitive to errors and noise. An alternative implementation that

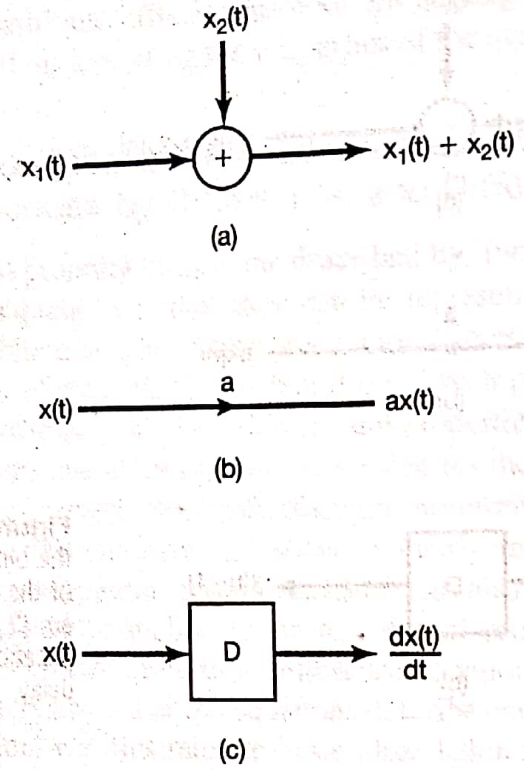


Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

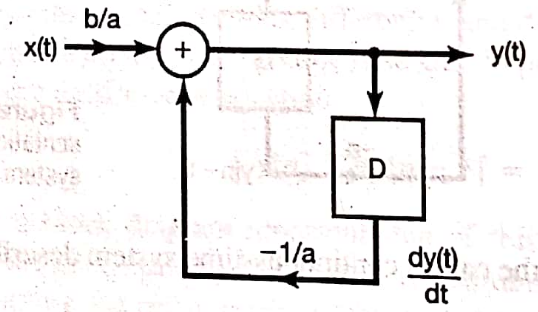


Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

is much more widely used can be obtained by first rewriting eq. (2.128) as

$$\frac{dy(t)}{dt} = bx(t) - ay(t) \tag{2.130}$$

and then integrating from $-\infty$ to t. Specifically, if we assume that the system described by eq. (2.130) is initially at rest, then the integral of $dy(t)/dt$ from $-\infty$ to t is precisely $y(t)$ (since the value of $y(-\infty)$ is zero). Consequently, we obtain the equation

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]\, d\tau. \tag{2.131}$$

In this form, our system can be implemented using the adder and coefficient multiplier indicated in Figure 2.29, together with an *integrator*, as defined in Figure 2.31. Figure 2.32 is a block diagram representation for this system using these elements.

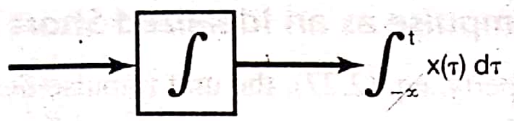


Figure 2.31 Pictorial representation of an integrator.

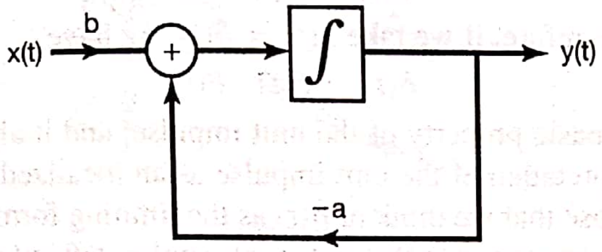


Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

Since integrators can be readily implemented using operational amplifiers, representations such as that in Figure 2.32 lead directly to analog implementations, and indeed, this is the basis for both early analog computers and modern analog computation systems. Note that in the continuous-time case it is the integrator that represents the memory storage element of the system. This is perhaps more readily seen if we consider integrating eq. (2.130) from a finite point in time t_0, resulting in the expression

$$y(t) = y(t_0) + \int_{t_0}^{t} [bx(\tau) - ay(\tau)]\, d\tau. \tag{2.132}$$

Equation (2.132) makes clear the fact that the specification of $y(t)$ requires an initial condition, namely, the value of $y(t_0)$. It is precisely this value that the integrator stores at time t_0.

While we have illustrated block diagram constructions only for the simplest first-order differential and difference equations, such block diagrams can also be developed for higher order systems, providing both valuable intuition for and possible implementations of these systems. Examples of block diagrams for higher order systems can be found in Problems 2.58 and 2.60.

2.5 SINGULARITY FUNCTIONS

In this section, we take another look at the continuous-time unit impulse function in order to gain additional intuitions about this important idealized signal and to introduce a set of related signals known collectively as *singularity functions*. In particular, in Section 1.4.2 we suggested that a continuous-time unit impulse could be viewed as the idealization of a pulse that is "short enough" so that its shape and duration is of no practical consequence—i.e., so that as far as the response of any particular LTI system is concerned, all of the area under the pulse can be thought of as having been applied instantaneously. In this section, we would first like to provide a concrete example of what this means and then use the interpretation embodied within the example to show that the key to the use of unit impulses and other singularity functions is in the specification of how LTI systems respond to these idealized signals; i.e., the signals are in essence defined in terms of how they behave under convolution with other signals.