Energy in Magnetic fields :—

It takes a certain amount of energy to start a current flowing in a circuit. Not considering resistive loss, but work one must do against back emf to get the current flow. This is fixed and recoverable part of energy.

Work done on unit charge against back emf, in one trip around circuit is $(-\mathcal{E})$ [-ve sign as work done by us against emf not work done by emf].

Amount of charge per unit time passing down wire is 'I'.

$\therefore$ work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E} I = L I \left(\frac{dI}{dt}\right).$$

Start with zero current, build up to final 'I' current

$$W = \text{work done} = \int^{I} L I \, dI = \frac{1}{2} L I^2$$

Doesn't depend on how long we take to crank up the current, only on geometry of loop (in the form of L) and final current I

flux through the loop

$$= \Phi = \oint_S B \cdot da = \int_S (\vec{\nabla} \times \vec{A}) \cdot da = \oint_p \vec{A} \cdot d\vec{l}$$

S ↳ surface boundary $\quad$ p ↳ parimeter of loop.

$$\therefore \Phi = L I = \oint A \cdot dl$$

$$W = \frac{1}{2} I \cdot L I = \frac{1}{2} I \cdot \oint \vec{A} \cdot d\vec{l} = \frac{1}{2} \oint (\vec{A} \cdot \vec{I}) \, dl$$

$$= \frac{1}{2} \int_V (\vec{A} \cdot \vec{J}) \, d\tau$$ $\quad$ '$\vec{J}$' is volume current or current per unit area.

But, $\vec{\nabla} \times B = \mu_0 \vec{J}$ (Amperes law)

$$\therefore W = \frac{1}{2} \int A \cdot \frac{(\vec{\nabla} \times \vec{B})}{\mu_0} \, d\tau$$

Now, $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

$$\therefore \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{B})$$
$$= \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

$$\therefore W = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \int_V \nabla\cdot(\vec{A}\times\vec{B})\, d\tau\right]$$

$$= \frac{1}{2\mu_0}\int_V \vec{B}^2 d\tau - \int_S (\vec{A}\times\vec{B})\cdot da$$

V → volume S → is surface boundary

∴ Larger the region we pick greater will be volume integral ∴ Smaller the surface integral [As surface gets further from current, both A and B decreases].

If we agree to integrate over all space

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau}$$

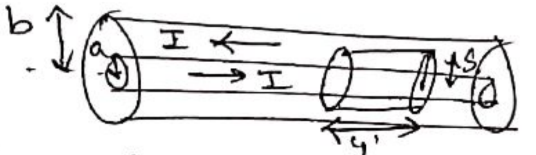
∴ (Energy is stored in magnetic field) is $\frac{B^2}{2\mu_0}$ per unit volume)

But energy stored in current distribution

$$W = \frac{1}{2}\int_V (\vec{A}\cdot\vec{J})\, d\tau$$

i.e $\frac{1}{2}(\vec{A}\cdot\vec{J})$ per unit volume.

Problem: A long coaxial cable carries current I (the current flowing down the surface of the inner cylinder) radius a, and back along the outer cylinder, radius 'b') Find the magnetic energy stored in a section of length 'l'.



From Amperes law, field betn cylinders

$$B = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad , \text{elsewhere field is zero}$$

Energy per unit volume

$$\frac{1}{2\mu_0}\cdot\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2} \qquad \left|\ \int \frac{1}{2\mu_0}B^2 d\tau\right.$$

Energy in a cylindrical shell of length 'l' radius 's', thickness 'ds'

$$= \left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right)\cdot\underbrace{(2\pi s\cdot l)\, ds}_{d\tau}$$

magnetic energy stored in section 'l'

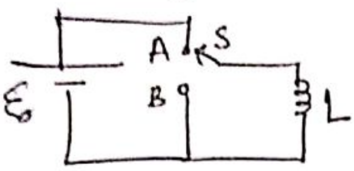
$$= \int_a^b \frac{\mu_0 I^2 l}{4\pi}\cdot\frac{ds}{s} = \frac{\mu_0 I^2 l}{4\pi}\ln(b/a)$$

As $W = \frac{1}{2\mu_0}\int_\tau B^2 d\tau = \frac{1}{2}LI^2 = \frac{\mu_0 I^2 l}{4\pi}\ln(b/a)$

$$\therefore \boxed{L = \frac{\mu_0 l}{2\pi}\ln(b/a)}$$

Q4) Problems on energy stored in mag. field

Suppose a circuit has been connected for a long time when suddenly, at time $t=0$, switch 'S' is thrown, bypassing battery,



i) What is the current at any subsequent time t?

ii) What is the total energy delivered to the resistor?

iii) Show that this is equal to energy originally stored in the inductor.

Solⁿ:- Initial current

$= I_0 = \mathcal{E}_0/R$, so $-L\left(\frac{dI}{dt}\right) = IR$

$\Rightarrow \frac{dI}{dt} = -\frac{IR}{L} \Rightarrow \int \frac{dI}{I} = \int -\frac{R\,dt}{L}$

$\Rightarrow I = I_0 \exp(-Rt/L) = \frac{\mathcal{E}_0}{R}\exp\left(\frac{-Rt}{L}\right)$

↳ Current at any time 't'

ii) Total energy per unit time delivered to resistor $= I^2R = R\left(\frac{\mathcal{E}_0}{R}\right)^2 \exp\left(\frac{-2Rt}{L}\right) = \left(\frac{dW}{dt}\right)$

$\therefore W = \frac{(\mathcal{E}_0)^2}{R} \int_0^{\infty} \exp\left(-\frac{2Rt}{L}\right) dt$

$= \frac{(\mathcal{E}_0)^2}{R} \cdot \frac{L}{2R}\left[\exp\left(-\frac{2Rt}{L}\right)\right]_0^{\infty}$

$= \frac{(\mathcal{E}_0)^2}{R} \times \frac{+L}{2R} = \frac{1}{2}L\left(\frac{\mathcal{E}_0}{R}\right)^2$ ⟸ energy delivered to resistor.

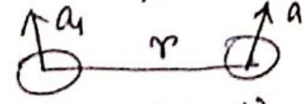
Same as energy stored in inductor

Q5) Two tiny wire loops with areas 'a_1' and 'a_2' are situated at a distance 'r' apart.

a) Find their mutual inductance.

b) If current 'I_1' flows in loop '1', we propose to turn on current 'I_2', in loop '2'. How much work must be done against mutually induced emf, to keep current 'I_1' flowing in loop 1?

Solⁿ a)



$B_1 = \frac{\mu_0}{4\pi}\cdot\frac{1}{r^3} I_1\left[3(a_1\cdot\hat{r})\hat{r} - a_1\right]$

Flux through loop 2

$= \Phi_2 = B_1\cdot a_2 = \left(\frac{\mu_0}{4\pi}\right)\frac{1}{r^3} I_1\left[3(\hat{a}_1\cdot\hat{r})(\hat{a}_2\cdot\hat{r}) - a_1\cdot a_2\right]$

$= M I_1$

$\therefore M = \frac{\mu_0}{4\pi r^3}\left[3(a_1\cdot\hat{r})(a_2\cdot\hat{r}) - a_1\cdot a_2\right]$ ✓

b) $\mathcal{E}_1 = -M\frac{dI_2}{dt}$, $\frac{dW}{dt} = -\mathcal{E}_1 I_1 = M I_1\left(\frac{dI_2}{dt}\right)$ ⟵ work done per unit time against mutual emf in loop 1 thus -ve sign.

As, I_1 = constant, $W_1 = M I_1 I_2$, $I_2 \rightarrow$ final current in loop 2, $W = \frac{\mu_0}{4\pi r^3}\left[3(m_1\cdot\hat{r})(m_2\cdot\hat{r}) - m_1\cdot m_2\right]$

Magnetic Energy (~~Energy stored in magnetic field~~) (In terms of ckt parameters, L, I etc)

Establishing a magnetic field requires the expenditure of energy. This conclusion follows directly from Faraday's law of induction: If source of voltage 'V' is applied to circuit, then, in general, the current through the ckt can be expressed by the law

$$V + \mathcal{E} = IR \quad \cdots (1)$$

'$\mathcal{E}$' is induced emf and R is resistance of current circuit. The work done by 'V' in moving the charge increment $dq = I\,dt$, through ckt is

$$V\,dq = VI\,dt = -\mathcal{E} I\,dt + I^2R\,dt$$

$$= I\,d\phi + I^2R\,dt$$

~~Where~~ As, $\frac{d\phi}{dt} = -\mathcal{E}$

The term $I^2R\,dt$ represents irreversible conversion of electrical energy into heat by the ckt.

But, 'I^2R' absorbs entire work input only in cases where the flux change is 'zero'.

Additional term '$I\,d\phi$' is work done against induced emf in the ckt. It is that part of work done by 'V' which is effective in altering the magnetic field structure.

∴ Disregarding 'I^2R' term, we write

$$dW_b = I\,d\phi$$

Subscript 'b' indicates that this work is performed by external electrical energy sources such as batteries. The work increment be either positive or negative. It is +ve when flux change '$d\phi$' through ckt is in the same direction as flux produced by current 'I'.

For rigid stationary ckt showing no energy losses other than Joule heat loss (ie no hysteresis), 'dW_b' = change in mag energy of the ckt.

Magnetic energy of Coupled Ckts

If there are n circuits, then electrical work done against induced emf is given by

$$dW_b = \sum_{i=1}^{n} I_i \, d\Phi_i \quad \cdots \cdots (1)$$

Above expression is perfectly general, it is valid independently how the flux increments '$d\Phi_i$' are produced.

We are interested in the case where '$d\Phi_i$' are produced by current changes in n - circuits themselves. Here, flux changes are directly correlated with changes in these currents.

$$d\Phi_i = \sum_{j=1}^{n} \left(\frac{d\Phi_{ij}}{dI_j} \right) dI_j = \sum_{j=1}^{n} M_{ij} \, dI_j \quad \cdots (2)$$

If ckts are rigid and stationary, then no mechanical work is associated with flux changes '$d\Phi_i$', and 'dW_b' is equal to change in magnetic energy 'dU' of the system.

Note that here we restrict our attention to stationary ckts. So magnetic energy can be calculated as work term.

Magnetic energy 'U' of a system of 'n' rigid stationary ckts obtained by integrating 'dW_b'.

From, zero flux situation (corresponding to $I_i = 0$) — to final flux value.

For a group of rigid ckts containing or located in linear magnetic media, 'Φ_i' are linearly related to currents in the ckt and magnetic energy is independent of the way in which these currents are brought to final set of values.

As final energy is independent of the order in which currents are varied, we may choose particular scheme for which 'W' is easily calculated. The scheme is all currents and all fluxes are brought to their final values in concert — i.e. at any instant of time all currents (and all fluxes) will be at the same fraction of their final values. Let us call fraction 'α', currents, $I_1, I_2 \cdots I_n$.

∴ At any stage

$I_i' = \alpha I_i$ and, $d\phi_i = \phi_i \, d\alpha$

$$\therefore \int dW_b = \int_0^1 d\alpha \sum_{i=1}^{n} I_i' \phi_i = \sum_{i=1}^{n} I_i \phi_i \int_0^1 \alpha \, d\alpha = \frac{1}{2} \sum_{i=1}^{n} I_i \phi_i$$

∴ Magnetic energy of 'n' coupled ckt

$$\boxed{U = \frac{1}{2} \sum_{i=1}^{n} I_i \phi_i}$$ ⇐ rigid ckt in linear media.

Similarly, From the eqn.

$$d\phi_i = \sum_{j=1}^{n} M_{ij} \, dI_j$$ for a rigid ckt, linear system can be integrated

$$\cancel{dW_b = \sum_{i=1}^{n} I_i \sum_{j=1}^{n} M_{ij} \, dI_j}$$

$$U = \int dW_b = \int \sum_{i=1}^{n} I_i \sum_{j=1}^{n} M_{ij} \, dI_j$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} I_i I_j$$

$$= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + \cdots + \frac{1}{2} L_n I_n^2$$
$$+ M_{12} I_1 I_2 + M_{13} I_1 I_3 + \cdots + M_{1n} I_1 I_n$$
$$+ M_{23} I_2 I_3 + \cdots + M_{n-1,n} I_{n-1} I_n.$$

rigid ckt linear medium

Here notation is $M_{ij} = M_{ji}$ and $M_{ii} = L_i$

For two coupled ckt.

$$\boxed{U = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2}$$

Energy density in the magnetic field

(Energy stored in magnetic field) itself. Can be extended to energy movement — through electromagnetic fields.

Consider a group of rigid current carrying ckt, none of which extends to infinity, immersed in a medium with linear magnetic properties.

U = Energy of such system $= \frac{1}{2}\sum_{i=1}^{n} I_i \Phi_i$

For present discussion it is convenient to assume that each ckt consists of only a single loop.

Flux may be expressed as,

$$\Phi_i = \int_{S_i} (\vec{B}\cdot\hat{n})\, da = \oint_{C_i} A\cdot dl_i$$

$A \rightarrow$ local vector potential

$$\therefore\ U = \frac{1}{2}\sum_{i=1}^{n}\oint_{C_i} I_i\, A\cdot dl_i$$

Making above expression more general, suppose we don't have current defined by wires, but instead each ckt is a closed path in the medium (assumed to be conducting) that follows a line of current density.

Thus 'U' may be made to approximate this situation very closely by choosing a large number of contiguous ckts (C_i)

Replacing $I_i\, dl_i \rightarrow J\, dv$

and substituting, $\int_V$ for $\sum_i \oint_{C_i}$

$$\therefore\ \boxed{U = \frac{1}{2}\int_V \vec{J}\cdot\vec{A}\, dv}$$

Above eqn can be transformed by field eqn $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = B\cdot(\vec{\nabla}\times\vec{A}) - A\cdot(\vec{\nabla}\times\vec{B})$$

$$\therefore\ U = \frac{1}{2}\int\left[\frac{(\vec{\nabla}\times\vec{B})}{\mu_0}\cdot\vec{A}\, dv\right]$$

$$= \frac{1}{2\mu_0}\int\left[\vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})\right] dv = \frac{1}{2\mu_0}\int_V \vec{B}\cdot(\vec{\nabla}\times\vec{A})\, dv - \int_S \frac{1}{2\mu_0}(\vec{A}\times\vec{B})\cdot n\, da$$

$$\therefore \boxed{U = \frac{1}{2\mu_0} \int_{v} B \cdot (\vec{\nabla} \times \vec{A})\, dv - \frac{1}{2\mu_0} \int_{S} (\vec{A} \times \vec{B}) \cdot \hat{n}\, da}$$

$S \rightarrow$ Surface which bounds volume 'v'.

By assumption, none of the current "circuits" extends to infinity, it is convenient to move the surface 'S' out to very large distance, so that all parts of the surface are far from currents.

The volume of the system must be increased accordingly. B falls ~~off~~ off at least as fast as $1/r^2$ 'r' is distance from origin near middle of current distribution to characteristic point on surface 'S'. A falls off at least as fast as $1/r$. Surface area is proportional to r^2

$\therefore$ Contribution from surface integral falls off as $1/r$ or faster, and if 'S' is moved out to infinity this contribution vanishes.

$$\therefore \boxed{U = \frac{1}{2\mu} \int_{v} (\vec{B} \cdot \vec{B})\, dv}, \quad \text{Since } \vec{B} = \vec{\nabla} \times \vec{A}$$

Above eqn is restricted to systems containing linear magnetic media

$\therefore$ energy density $= u = \frac{1}{2} \cdot \vec{H} \cdot \vec{B}$

$\therefore$ For isotropic, linear, magnetic material

$$\boxed{u = \frac{1}{2} \mu H^2 = \frac{B^2}{2\mu}}$$