CHAPTER 8

Fourier Series and Fourier Transform of Discrete Time Signals

8.1 Introduction

A periodic discrete time signal with fundamental period N can be decomposed into N harmonically related frequency components. The summation of the frequency components gives the Fourier series representation of periodic discrete time signal, in which the discrete time signal is represented as a function of frequency, ω. The Fourier series of discrete time signal is called ***Discrete Time Fourier Series*** (***DTFS***). The frequency components are also called frequency spectrum of the discrete time signal.

The Fourier representation of periodic discrete time signals has been extended to nonperiodic signals by letting the fundamental period N to infinity, and this Fourier method of representing nonperiodic discrete time signals as a function of discrete time frequency, ω is called Fourier transform of discrete time signals or ***Discrete Time Fourier Transform*** (***DTFT***). The Fourier represention of discrete time signals is also known as frequency domain representation. In general the Fourier series representation can be obtained only for periodic discrete time signals, but the Fourier transform technique can be applied to both periodic and nonperiodic signals to obtain the frequency domain representation of the discrete time signals.

The Fourier representation of discrete time signals can be used to perform frequency domain analysis of discrete time signals, in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components. The graphical plots of magnitude and phase as a function of frequency are also drawn. The plot of magnitude versus frequency is called ***magnitude spectrum*** and the plot of phase versus frequency is called ***phase spectrum***. In general these plots are called ***frequency spectrum***.

8.2 Fourier Series of Discrete Time Signals (Discrete Time Fourier Series)

The Fourier series (or ***Discrete Time Fourier Series***, DTFS) of discrete time periodic signal x(n) with periodicity N is defined as,

$$x(n) = \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} c_k \, e^{j\omega_0 kn} = \sum_{k=0}^{N-1} c_k \, e^{j\omega_k n} \quad .....(8.1)$$

where, c_k = Fourier coefficients; ω_0 = Fundamental frequency of x(n)

$$\omega_k = \frac{2\pi k}{N} = k^{th} \text{ harmonic frequency of } x(n)$$

$$c_k e^{j\omega_k n} = k^{th} \text{ harmonic component of } x(n)$$

The Fourier coefficients, c_k for k = 0, 1, 2, ..., N–1 can be evaluated using equation (8.2).

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, ..., N-1 \qquad .....(8.2)$$

The ***Fourier coefficient*** c_k represents the amplitude and phase associated with the k^{th} frequency component. Hence we can say that the fourier coefficients provide the description of x(n) in the frequency domain.

Proof :

Consider the Fourier series representation of the discrete time signal x(n).

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

Let us replace k by p

$$\therefore\ x(n) = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}$$

Let us multiply the above equation by $e^{\frac{-j2\pi kn}{N}}$ on both sides.

$$x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

On evaluating the above equation for n = 0 to N – 1 and summing up the values we get,

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

Let us interchange the order of summation in the right hand side of the above equation and rearrange as shown below.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}}$$

When p = k the right hand side of the above equation reduces to c_k N.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = c_k N$$

$$\therefore\ c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Note : The sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

$$\therefore\ \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}} = N\ ;\ (p-k) = 0, \pm N, \pm 2N, ...$$
$$= 0\ ;\ (p-k) \neq N$$

Difference Between Continuous Time and Discrete Time Fourier Series

1. The frequency range of continuous time signal is $-\infty$ to $+\infty$, and so it has infinite frequency spectrum.
2. The frequency range of discrete time signal is 0 to 2π (or $-\pi$ to $+\pi$) and so it has finite frequency spectrum. A discrete time signal with fundamental period N will have N frequency components whose frequencies are,

$$\omega_k = \frac{2\pi k}{N} \quad ; \text{ for } k = 0, 1, 2, ... , N-1$$

8.2.1 Frequency Spectrum of Periodic Discrete Time Signals

Let x(n) be a periodic discrete time signal. Now, the Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}}$$

where, c_k is the Fourier coefficient of k^{th} harmonic component

The Fourier coefficient, c_k is a complex quantity and so it can be expressed in the polar form as shown below.

$$c_k = |c_k| \angle c_k \; ; \qquad \text{for } k = 0, 1, 2, 3, \ldots\ldots N-1$$

where, $|c_k|$ = Magnitude of c_k ; $\angle c_k$ = Phase of c_k

The term, $|c_k|$ represents the magnitude of k^{th} harmonic component and the term $\angle c_k$ represents the phase of the k^{th} harmonic component.

The plot of harmonic magnitude / phase of a discrete time signal versus "k" (or harmonic frequency ω_k) is called ***Frequency spectrum***. The plot of harmonic magnitude versus "k" (or ω_k) is called ***magnitude spectrum*** and the plot of harmonic phase versus "k" (or ω_k) is called ***phase spectrum***.

The Fourier coefficients are periodic with period N.

$$\therefore c_{k+N} = c_k$$

Since Fourier coefficients are periodic, the frequency spectrum is also periodic, with period N.

Proof :

Consider the Fourier coefficient c_k of the discrete time signal x(n).

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Now, the Fourier coefficient c_{k+N} is given by,

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi (k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\left(\frac{-j2\pi kn}{N} + \frac{-j2\pi Nn}{N}\right)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}\, e^{-j2\pi n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = c_k$$

For a periodic discrete time signal with period N, there are N Fourier coefficients denoted as c_0, c_1, c_2, ... c_{N-1}, and so the N-number of Fourier coefficients can be expressed as a sequence consisting of N values.

Fourier coefficients,	c_k	=	$\{c_0, c_1, c_2, c_3, \ldots\ldots\ldots c_{N-1}\}$
Magnitude spectrum,	$\lvert c_k\rvert$	=	$\{\lvert c_0\rvert, \lvert c_1\rvert, \lvert c_2\rvert, \lvert c_3\rvert, \ldots\ldots\ldots \lvert c_{N-1}\rvert\}$
Phase spectrum,	$\angle c_k$	=	$\{\angle c_0, \angle c_1, \angle c_2, \angle c_3, \ldots\ldots\ldots \angle c_{N-1}\}$

8.2.2 Properties of Discrete Time Fourier Series

The properties of discrete time Fourier series coefficients are listed in table 8.1. The proof of these properties are left as exercise to the readers.

Table 8.1 : Properties of Discrete Time Fourier Series Coefficients

Note : c_k are Fourier series coefficients of x(n) and d_k are Fourier series coefficients of y(n).

Property	Discrete time periodic signal	Fourier series coefficients
Linearity	$A\,x(n) + B\,y(n)$	$A\,c_k + B\,d_k$
Time shifting	$x(n-m)$	$c_k\, e^{\frac{j2\pi km}{N}}$
Frequency shifting	$e^{\frac{j2\pi nm}{N}}\, x(n)$	c_{k-m}
Conjugation	$x^*(n)$	c^*_{-k}
Time reversal	$x(-n)$	c_{-k}
Time scaling	$x(\frac{n}{m})$; for n multiple of m (periodic with period mN)	$\frac{1}{m}\, c_k$
Multiplication	$x(n)\, y(n)$	$\sum_{m=0}^{N-1} c_m\, d_{k-m}$
Circular convolution	$\sum_{m=0}^{N-1} x(m)\, y((n-m))_N$	$N\, c_K\, d_K$
Symmetry of real signals	x(n) is real	$c_k = c^*_{-k}$ $\lvert c_k\rvert = \lvert c_{-k}\rvert$ $\angle c_k = -\angle c_{-k}$ $Re\{c_k\} = Re\{c_{-k}\}$ $Im\{c_k\} = -Im\{c_{-k}\}$
Real and even	x(n) is real and even	c_k are real and even
Real and odd	x(n) is real and odd	c_k are imaginary and odd
Parseval's relation	Average power P of x(n) is defined as, $P = \frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2$	Average power P in terms of Fourier series coefficients is, $P = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$
	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$	

*Note : The average power in the signal is the sum of the powers of the individual frequency components. The sequence $|c_k|^2$ for k = 0, 1, 2,....., (N − 1) is the distribution of power as a function of frequency and so it is called the **power density spectrum** (or) **power spectral density** of the periodic signal.*

Example 8.1

Determine the Fourier series representation of the following discrete time signals.

a) $x(n) = 2\cos\sqrt{3}\pi n$ b) $x(n) = 4\cos\frac{\pi n}{2}$ c) $x(n) = 3e^{\frac{j5\pi n}{2}}$

Solution

a) Given that, $x(n) = 2\cos\sqrt{3}\pi n$

Test for Periodicity

Let, $x(n+N) = 2\cos\sqrt{3}\pi(n+N) = 2\cos\left(\sqrt{3}\pi n + \sqrt{3}\pi N\right)$

For periodicity $\sqrt{3}\pi N$ should be equal to integral multiple of 2π.

Let, $\sqrt{3}\pi N = M \times 2\pi$; where M and N are integers. $\Rightarrow$ $N = \frac{2}{\sqrt{3}}M$

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

Fourier Series

Here x(n) is nonperiodic signal and so Fourier series does not exists.

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b) Given that, $x(n) = 4\cos\frac{\pi n}{2}$

Test for Periodicity

Let, $x(n+N) = 4\cos\frac{\pi}{2}(n+N) = 4\cos\left(\frac{\pi n}{2} + \frac{\pi N}{2}\right)$

For periodicity $\frac{\pi N}{2}$ should be integral multiple of 2π.

Let, $\frac{\pi N}{2} = 2\pi \times M$; where M and N are integers $\Rightarrow$ $N = 4M$

Here N is an integer for M = 1, 2, 3, ...

Let M = 1, $\therefore$ N = 4

Hence x(n) is periodic, with fundamental period N = 4, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$.

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, 3, \ldots, N-1$$

Here N = 4 and $x(n) = 4\cos\frac{\pi n}{2}$

$$\therefore c_k = \frac{1}{4}\sum_{n=0}^{3} 4\cos\frac{\pi n}{2}\, e^{\frac{-j2\pi kn}{4}} \quad ; \text{ for } k = 0, 1, 2, 3$$

$$= \frac{4}{4}\sum_{n=0}^{3}\cos\frac{\pi n}{2}\, e^{\frac{-j\pi kn}{2}} = \sum_{n=0}^{3}\cos\frac{\pi n}{2}\left(\cos\frac{\pi kn}{2} - j\sin\frac{\pi kn}{2}\right)$$

$\cos 0 = 1$;	$\cos\pi = -1$
$\cos\frac{\pi}{2} = 0$;	$\cos\frac{3\pi}{2} = 0$

$$= \cos 0\,(\cos 0 - j\sin 0) + \cos\frac{\pi}{2}\left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) + \cos\pi\,(\cos\pi k - j\sin\pi k) + \cos\frac{3\pi}{2}\left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$= 1 + 0 - (\cos\pi k - j\sin\pi k) + 0 = 1 - \cos\pi k + j\sin\pi k$$

When k = 0 ; $c_k = c_0 = 1 - \cos 0 + j\sin 0 \quad = 1 - 1 + j0 = 0$

When k = 1 ; $c_k = c_1 = 1 - \cos \pi + j\sin \pi \quad = 1 + 1 + j0 = 2$

When k = 2 ; $c_k = c_2 = 1 - \cos 2\pi + j\sin 2\pi = 1 - 1 + j0 = 0$

When k = 3 ; $c_k = c_3 = 1 - \cos 3\pi + j\sin 3\pi = 1 + 1 + j0 = 2$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}} = c_0 + c_1 e^{\frac{j\pi n}{2}} + c_2 e^{j\pi n} + c_3 e^{\frac{j3\pi n}{2}}$$

$$= 0 + 2e^{\frac{j\pi n}{2}} + 0 + 2e^{\frac{j3\pi n}{2}} = 2e^{\frac{j\pi n}{2}} + 2e^{\frac{j3\pi n}{2}} = 2e^{j\omega_0 n} + 2e^{j3\omega_0 n} \;;\quad \text{where } \omega_0 = \frac{\pi}{2}$$

c) Given that, $x(n) = 3e^{\frac{j5\pi n}{2}}$

Test for Periodicity

Let, $x(n+N) = 3e^{\frac{j5\pi(n+N)}{2}} = 3e^{\left(\frac{j5\pi n}{2} + \frac{j5\pi N}{2}\right)}$

For periodicity $\frac{5\pi N}{2}$ should be integral multiple of 2π.

Let, $\frac{5\pi N}{2} = 2\pi \times M \quad \Rightarrow \quad N = \frac{4}{5}M$

Here N is integer for M = 5, 10, 15, ...

Let, M = 5, ∴ N = 4

Here x(n) is periodic with fundamental period N = 4, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \;;\quad \text{for } k = 0, 1, 2, 3, \ldots, N-1$$

Here N = 4 and $x(n) = 3e^{\frac{j5\pi n}{2}}$

$$\therefore c_k = \frac{1}{4}\sum_{n=0}^{3} 3e^{\frac{j5\pi n}{2}}\, e^{\frac{-j2\pi kn}{4}} \;;\quad \text{for } k = 0, 1, 2, 3$$

$$= \frac{3}{4}\sum_{n=0}^{3} e^{\frac{j\pi n(5-k)}{2}} = \frac{3}{4}\left[e^0 + e^{\frac{j\pi(5-k)}{2}} + e^{\frac{j2\pi(5-k)}{2}} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{3}{4}\left[1 + e^{\frac{j\pi(5-k)}{2}} + e^{j\pi(5-k)} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{3}{4}\left[1 + \cos\frac{\pi(5-k)}{2} + j\sin\frac{\pi(5-k)}{2} + \cos\pi(5-k) + j\sin\pi(5-k) + \cos\frac{3\pi(5-k)}{2} + j\sin\frac{3\pi(5-k)}{2}\right]$$

When k = 0; $c_k = c_0 = \frac{3}{4}\left[1 + \cos\frac{5\pi}{2} + j\sin\frac{5\pi}{2} + \cos 5\pi + j\sin 5\pi + \cos\frac{15\pi}{2} + j\sin\frac{15\pi}{2}\right]$

$$= \frac{3}{4}[1 + 0 + j - 1 + j0 + 0 - j] = 0$$

When $k = 1;\ c_k = c_1 = \frac{3}{4}\left[1+\cos 2\pi + j\sin 2\pi + \cos 4\pi + j\sin 4\pi + \cos 6\pi + j\sin 6\pi\right]$

$= \frac{3}{4}\left[1 + 1 + j0 + 1 + j0 + 1 + j0\right] = 3$

When $k = 2;\ c_k = c_2 = \frac{3}{4}\left[1 + \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} + \cos 3\pi + j\sin 3\pi + \cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}\right]$

$= \frac{3}{4}\left[1 + 0 - j - 1 + j0 + 0 + j\right] = 0$

When $k = 3;\ c_k = c_3 = \frac{3}{4}\left[1+\cos\pi + j\sin\pi + \cos 2\pi + j\sin 2\pi + \cos 3\pi + j\sin 3\pi\right]$

$= \frac{3}{4}\left[1 - 1 + j0 + 1 + j0 - 1 + j0\right] = 0$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}}$$

$$= c_0 + c_1 e^{\frac{j\pi n}{2}} + c_2 e^{j\pi n} + c_3 e^{\frac{j3\pi n}{2}} = 0 + 3e^{\frac{j\pi n}{2}} + 0 + 0 = 3e^{\frac{j\pi n}{2}} = 3e^{j\omega_0 n}$$

> ***Note*** *:* $x(n) = 3e^{\frac{j5\pi n}{2}} = 3e^{j\left(\frac{4\pi n}{2} + \frac{\pi n}{2}\right)} = 3e^{j2\pi n}\, e^{\frac{j\pi n}{2}} = 3e^{\frac{j\pi n}{2}} = 3e^{j\omega_0 n}$
>
> $\therefore$ *The given signal itself is in the Fourier series form.*

Example 8.2

Determine the Fourier series representation of the following discrete time signal and sketch the frequency spectrum.

$$x(n) = \{\ldots.., 1, 2, -1, \underset{\uparrow}{1}, 2, -1, 1, 2, -1, \ldots..\}$$

Solution

Given that, $x(n) = \{\ldots.., 1, 2, -1, \underset{\uparrow}{1}, 2, -1, 1, 2, -1, \ldots..\}$

Here x(n) is periodic with periodicity of N = 3, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$.

Let, x(n) = {1, 2, –1} (considering one period). Now, the Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \frac{1}{3}\sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[x(0) + x(1)\, e^{\frac{-j2\pi k}{3}} + x(2)\, e^{\frac{-j4\pi k}{3}}\right] = \frac{1}{3}\left[1 + 2e^{\frac{-j2\pi k}{3}} - e^{\frac{-j4\pi k}{3}}\right]$$

When $k = 0;\ c_k = c_0 = \frac{1}{3}\left[1 + 2 - 1\right] = \frac{2}{3} = 0.667$

When $k = 1;\ c_k = c_1 = \frac{1}{3}\left[1 + 2e^{\frac{-j2\pi}{3}} - e^{\frac{-j4\pi}{3}}\right]$

$= \frac{1}{3}\left[1 + 2\cos\frac{2\pi}{3} - j2\sin\frac{2\pi}{3} - \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3}\right]$

$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} - j2\times\frac{\sqrt{3}}{2} + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right]$

$= \frac{1}{3}\left[\frac{1}{2} - j\frac{3\sqrt{3}}{2}\right] = \frac{1}{6} - j\frac{\sqrt{3}}{2} = 0.1667 - j0.866$

$= 0.88\angle -1.38\ \text{rad} = 0.88\angle -0.44\pi = 0.88\, e^{-j0.44\pi}$

When k = 2; $c_k = c_2 = \frac{1}{3}\left[1 + 2e^{\frac{-j4\pi}{3}} - e^{\frac{-j8\pi}{3}}\right]$

$$= \frac{1}{3}\left[1 + 2\cos\frac{4\pi}{3} - j2\sin\frac{4\pi}{3} - \cos\frac{8\pi}{3} + j\sin\frac{8\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} + j2\times\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{1}{2} + j\frac{3\sqrt{3}}{2}\right] = \frac{1}{6} + j\frac{\sqrt{3}}{2} = 0.1667 + j0.866$$

$$= 0.88\angle 1.38\text{ rad} = 0.88\angle 0.44\pi = 0.88\,e^{j0.44\pi}$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{2} c_k\, e^{\frac{j2\pi kn}{3}}$$

$$= c_0 + c_1 e^{\frac{j2\pi n}{3}} + c_2 e^{\frac{j4\pi n}{3}}$$

$$= 0.667 + 0.88\,e^{-j0.44\pi}\, e^{\frac{j2\pi n}{3}} + 0.88\,e^{j0.44\pi}\, e^{\frac{j4\pi n}{3}}$$

$$= 0.667 + 0.88\,e^{-j0.44\pi}\, e^{j\omega_0 n} + 0.88\,e^{j0.44\pi}\, e^{j2\omega_0 n}$$

Frequency Spectrum

The frequency spectrum has two components : Magnitude spectrum and Phase spectrum.

The magnitude spectrum is obtained from magnitude of c_k and phase spectrum is obtained from phase of c_k.

Here, $c_k = \{c_0,\ c_1,\ c_2\} = \{0.667,\ 0.88\angle -0.44\pi,\ 0.88\angle 0.44\pi\}$

$\therefore$ Magnitude spectrum, $|c_k| = \{0.667,\ 0.88,\ 0.88\}$

Phase spectrum, $\angle c_k = \{0,\ -0.44\pi,\ 0.44\pi\}$

The sketch of magnitude and phase spectrum are shown in fig 1.

Here both the spectrum are periodic with period, N = 3.

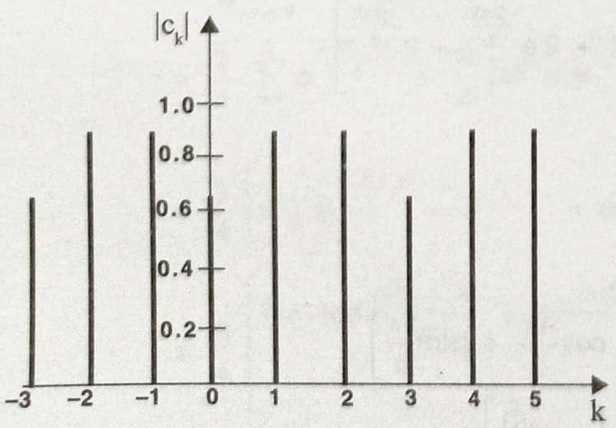

Fig 1.a : Magnitude spectrum.

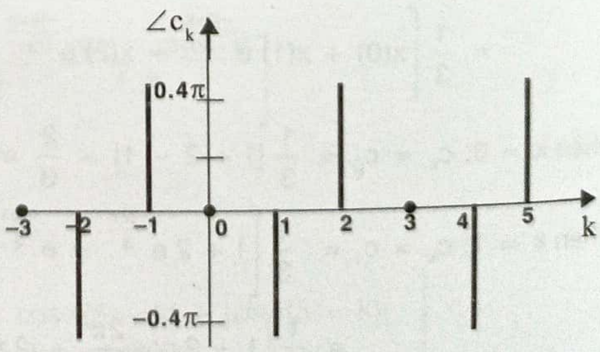

Fig 1.b : Phase spectrum.

Fig 1 : Frequency spectrum.