Fourier Series and Fourier Transform of Continuous Time Signals

4.1 Introduction

The French mathematician Jean Baptiste Joseph Fourier (J.B.J. Fourier) has shown that any periodic non-sinusoidal signal can be expressed as a linear weighted sum of harmonically related sinusoidal signals. This leads to a method called ***Fourier series*** in which a periodic signal is represented as a function of frequency.

The Fourier representation of periodic signals has been extended to non-periodic signals by letting the fundamental period T tend to infinity, and this Fourier method of representing non-periodic signals as a function of frequency is called ***Fourier transform***. The Fourier represention of signals is also known as frequency domain representation. In general, the Fourier series representation can be obtained only for periodic signals, but the Fourier transform technique can be applied to both periodic and non-periodic signals to obtain the frequency domain representation of the signals.

The Fourier representation of signals can be used to perform frequency domain analysis of signals, in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components. The graphical plots of magnitude and phase as a function of frequency are also drawn. The plot of magnitude versus frequency is called ***magnitude spectrum*** and the plot of phase versus frequency is called ***phase spectrum***. In general, these plots are called ***frequency spectrum***.

4.2 Trigonometric Form of Fourier Series

4.2.1 Definition of Trigonometric Form of Fourier Series

The ***trigonometric form of Fourier series*** of a periodic signal, x(t), with period T is defined as,

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t \qquad .....(4.1)$$

$$\therefore x(t) = \frac{1}{2}a_0 + a_1 \cos\Omega_0 t + a_2 \cos 2\Omega_0 t + a_3 \cos 3\Omega_0 t + ...$$
$$+ b_1 \sin\Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + ...$$

where, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$ = Fundamental frequency in rad/sec

F_0 = Fundamental frequency in cycles/sec or Hz

$n\Omega_0$ = Harmonic frequencies

a_0, a_n, b_n = Fourier coefficients of trigonometric form of Fourier series

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The Fourier coefficients can be evaluated using the following formulae.

> **Note :** *1. Here $a_0/2$ is the value of constant component of the signal x(t).*
> *2. The Fourier coefficient a_n and b_n are maximum amplitudes of n^{th} harmonic component*

The **Fourier coefficients** can be evaluated using the following formulae.

$$a_0 = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{2}{T}\int_{0}^{T} x(t)\,dt \qquad .....(4.2)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{2}{T}\int_{0}^{T} x(t)\cos n\Omega_0 t\,dt \qquad .....(4.3)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin n\Omega_0 t\,dt \quad \text{(or)} \quad b_n = \frac{2}{T}\int_{0}^{T} x(t)\sin n\Omega_0 t\,dt \qquad .....(4.4)$$

In the above formulae, the limits of integration are either **–T/2 to +T/2** or **0 to T**. In general, the limit of integration is one period of the signal and so the limits can be from $\mathbf{t_0}$ **to** $\mathbf{t_0 + T}$, where t_0 is any time instant.

4.2.2 Conditions for Existence of Fourier Series

The Fourier series exists only if the following Dirichlet's conditions are satisfied.

1. The signal x(t) is well defined and single valued, except possibly at a finite number of points.
2. The signal x(t) must possess only a finite number of discontinuities in the period T.
3. The signal must have a finite number of positive and negative maxima in the period T.

> **Note :** *1. The value of signal x(t) at $t = t_0$ is $x(t_0)$ if $t = t_0$ is a point of continuity.*
> *2. The value of signal x(t) at $t = t_0$ is $\dfrac{x(t_0^+) + x(t_0^-)}{2}$ if $t = t_0$ is a point of discontinuity.*

4.2.3 Derivation of Equations for a_0, a_n and b_n

Evaluation of a_0

The Fourier coefficient a_0 is given by,

$$a_0 = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{2}{T}\int_{0}^{T} x(t)\,dt$$

Proof :

Consider the trigonometric Fourier series of x(t), (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\,dt = \int_0^T \frac{a_0}{2}\,dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t\,dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t\,dt$$

$$= \frac{a_0}{2}\int_0^T dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\Omega_0 t\,dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin n\Omega_0 t\,dt$$

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$$\therefore \int_0^T x(t)\,dt = \frac{a_0}{2}[t]_0^T + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^T + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_0^T$$

$$= \frac{a_0}{2}[T-0] + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\Omega_0 T}{n\Omega_0} - \frac{\sin 0}{n\Omega_0}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n\Omega_0 T}{n\Omega_0} + \frac{\cos 0}{n\Omega_0}\right]$$

$$= \frac{T}{2}a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{1}{n\frac{2\pi}{T}}\right]$$

$$= \frac{T}{2}a_0 + \sum_{n=1}^{\infty} a_n T\left[\frac{\sin n2\pi}{n2\pi}\right] + \sum_{n=1}^{\infty} b_n T\left[\frac{-\cos n2\pi}{n2\pi} + \frac{1}{n2\pi}\right]$$

$$= \frac{T}{2}a_0 + \sum_{n=1}^{\infty} a_n T \times 0 + \sum_{n=1}^{\infty} b_n T\left[-\frac{1}{n2\pi} + \frac{1}{n2\pi}\right] = \frac{T}{2}a_0$$

$$\therefore a_0 = \frac{2}{T}\int_0^T x(t)\,dt$$

$\Omega_0 = \frac{2\pi}{T}$
$\sin 0 = 0$
$\cos 0 = 1$

$\sin n2\pi = 0$; $\cos n2\pi = 1$ for integer n

Evaluation of a_n

The Fourier coefficient a_n is given by,

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt$$

Proof :

Consider the trigonometric form of Fourier series of x(t), (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + \ldots + a_k\cos k\Omega_0 t + \ldots$$
$$+ b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + \ldots + b_k\sin k\Omega_0 t + \ldots$$

Let us multiply the above equation by $\cos k\Omega_0 t$

$$\therefore x(t)\cos k\Omega_0 t = \frac{a_0}{2}\cos k\Omega_0 t + a_1\cos\Omega_0 t\cos k\Omega_0 t + a_2\cos 2\Omega_0 t\cos k\Omega_0 t + \ldots$$
$$\ldots + a_k\cos^2 k\Omega_0 t + \ldots + b_1\sin\Omega_0 t\cos k\Omega_0 t + b_2\sin 2\Omega_0 t\cos k\Omega_0 t + \ldots$$
$$\ldots + b_k\sin k\Omega_0 t\cos k\Omega_0 t + \ldots$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\cos k\Omega_0 t\,dt = \int_0^T \frac{a_0}{2}\cos k\Omega_0 t\,dt + \int_0^T a_1\cos\Omega_0 t\cos k\Omega_0 t\,dt$$
$$+ \int_0^T a_2\cos 2\Omega_0 t\cos k\Omega_0 t\,dt + \ldots + \int_0^T a_k\cos^2 k\Omega_0 t\,dt + \ldots + \int_0^T b_1\sin\Omega_0 t\cos k\Omega_0 t\,dt$$
$$+ \int_0^T b_2\sin 2\Omega_0 t\cos k\Omega_0 t\,dt + \ldots + \int_0^T b_k\sin k\Omega_0 t\cos k\Omega_0 t\,dt + \ldots \quad \ldots(4.5)$$

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$$\therefore \int_0^T x(t)\cos k\Omega_0 t\, dt = \int_0^T a_k \cos^2 k\Omega_0 t\, dt = a_k \int_0^T \frac{1+\cos 2k\Omega_0 t}{2}\, dt$$

$$= \frac{a_k}{2}\int_0^T (1+\cos 2k\Omega_0 t)\, dt = \frac{a_k}{2}\left[t + \frac{\sin 2k\Omega_0 t}{2k\Omega_0}\right]_0^T$$

$$= \frac{a_k}{2}\left[T + \frac{\sin 2k\Omega_0 T}{2k\Omega_0} - 0 - \frac{\sin 0}{2k\Omega_0}\right] = \frac{a_k}{2}\left[T + \frac{\sin 2k\frac{2\pi}{T}T}{2k\frac{2\pi}{T}}\right] = \frac{T}{2}a_k$$

In equation (4.5) all definite integrals will be zero, except $\int_0^T a_k \cos^2 k\Omega_0 t\, dt$.

$\Omega_0 = \frac{2\pi}{T}$

$\sin 0 = 0$

$\sin 2k2\pi = 0$ for integer k

$$\therefore a_k = \frac{2}{T}\int_0^T x(t)\cos k\Omega_0 t\, dt \quad .....(4.6)$$

The equation (4.6) gives the k[th] coefficient a_k. Hence the n[th] coefficient a_n is given by,

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\, dt$$

Evaluation of b_n

The Fourier coefficient b_n is given by,

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin n\Omega_0 t\, dt \quad \text{(or)} \quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt$$

Proof :

Consider the trigonometric form of Fourier series of x(t) (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + ... + a_k\cos k\Omega_0 t + ...$$
$$+ b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + ... + b_k\sin k\Omega_0 t + ...$$

Let us multiply the above equation by $\sin k\Omega_0 t$

$$\therefore x(t)\sin k\Omega_0 t = \frac{a_0}{2}\sin k\Omega_0 t + a_1\cos\Omega_0 t\sin k\Omega_0 t + a_2\cos 2\Omega_0 t\sin k\Omega_0 t + ...$$
$$..... + a_k\cos k\Omega_0 t\sin k\Omega_0 t + ... + b_1\sin\Omega_0 t\sin k\Omega_0 t$$
$$+ b_2\sin 2\Omega_0 t\sin k\Omega_0 t + ... + b_k\sin^2 k\Omega_0 t + ...$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\sin k\Omega_0 t\, dt = \int_0^T \frac{a_0}{2}\sin k\Omega_0 t\, dt + \int_0^T a_1\cos\Omega_0 t\sin k\Omega_0 t\, dt$$
$$+ \int_0^T a_2\cos 2\Omega_0 t\sin k\Omega_0 t\, dt + ... + \int_0^T a_k\cos k\Omega_0 t\sin k\Omega_0 t\, dt + ...$$
$$..... + \int_0^T b_1\sin\Omega_0 t\sin k\Omega_0 t\, dt + \int_0^T b_2\sin 2\Omega_0 t\sin k\Omega_0 t\, dt + ...$$
$$+ \int_0^T b_k\sin^2 k\Omega_0 t\, dt + ... \quad .....(4.7)$$

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$$\therefore \int_0^T x(t)\sin k\Omega_0 t\,dt = \int_0^T b_k \sin^2 k\Omega_0 t\,dt = b_k \int_0^T \frac{1-\cos 2k\Omega_0 t}{2}\,dt$$

$$= \frac{b_k}{2}\int_0^T (1-\cos 2k\Omega_0 t)\,dt = \frac{b_k}{2}\left[t - \frac{\sin 2k\Omega_0 t}{2k\Omega_0}\right]_0^T$$

$$= \frac{b_k}{2}\left[T - \frac{\sin 2k\Omega_0 T}{2k\Omega_0} - 0 + \frac{\sin 0}{2k\Omega_0}\right]$$

$$= \frac{b_k}{2}\left[T - \frac{\sin 2k\frac{2\pi}{T}T}{2k\frac{2\pi}{T}}\right] = \frac{T}{2}b_k$$

In equation (4.7) all definite integrals will be zero, except $\int_0^T b_k \sin^2 k\Omega_0 t\,dt$.

$\Omega_0 = \frac{2\pi}{T}$

$\sin 0 = 0$

$\sin 2k2\pi = 0$ for integer k

$$\therefore b_k = \frac{2}{T}\int_0^T x(t)\sin k\Omega_0 t\,dt \qquad .....(4.8)$$

The equation (4.8) gives k^{th} coefficient b_k. Hence the n^{th} coefficient b_n is given by,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt$$

4.3 Exponential Form of Fourier Series

4.3.1 Definition of Exponential Form of Fourier Series

The ***exponential form of Fourier series*** of a periodic signal x(t) with period T is defined as,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} \qquad .....(4.9)$$

where, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$ = Fundamental frequency in rad/sec

F_0 = Fundamental frequency in cycles/sec or Hz

$\pm n\Omega_0$ = Harmonic frequencies

c_n = Fourier coefficients of exponential form of Fourier series.

The ***Fourier coefficient*** c_n can be evaluated using the following equation.

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,e^{-jn\Omega_0 t}\,dt \quad \text{(or)} \quad c_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\Omega_0 t}\,dt \qquad .....(4.10)$$

In equation (4.10), the limits of integration are either **–T/2 to +T/2** or **0 to T**. In general, the limit of integration is one period of the signal and so the limits can be from t_0 **to** $t_0 + T$, where t_0 is any time instant.

4.3.2 Negative Frequency

The exponential form of Fourier series representation of a signal x(t) has complex exponential harmonic components for both positive and negative frequencies. When the positive and negative complex exponential components of same harmonic are added, it gives rise to real sine or cosine signals.

Alternatively, when the real sine or cosine signal has to be represented in terms of complex exponential then a signal with negative frequency is required.

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Here it should be understood that the signal with negative frequency is not a physically realizable signal, but it is required for mathematical representation of real signals in terms of complex exponential signals.

4.3.3 Derivation of Equation for c_n

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \quad \text{(or)} \quad c_n = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-jn\Omega_0 t}\, dt$$

Proof :

Consider the exponential form of Fourier series of x(t), (equation (4.9)).

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} = \ldots + c_{-k} e^{-jk\Omega_0 t} + \ldots + c_{-2} e^{-j2\Omega_0 t} + c_{-1} e^{-j\Omega_0 t} + c_0 + c_1 e^{j\Omega_0 t} + c_2 e^{j2\Omega_0 t} + \ldots + c_k e^{jk\Omega_0 t} + \ldots$$

Let us multiply the above equation by $e^{-jk\Omega_0 t}$.

$$\therefore x(t) e^{-jk\Omega_0 t} = \ldots + c_{-k} e^{-j2k\Omega_0 t} + \ldots + c_{-2} e^{-j2\Omega_0 t} e^{-jk\Omega_0 t} + c_{-1} e^{-j\Omega_0 t} e^{-jk\Omega_0 t} + c_0 e^{-jk\Omega_0 t} + c_1 e^{j\Omega_0 t} e^{-jk\Omega_0 t} + c_2 e^{j2\Omega_0 t} e^{-jk\Omega_0 t} + \ldots + c_k + \ldots$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t) e^{-jk\Omega_0 t} dt = \ldots + \int_0^T c_{-k} e^{-j2k\Omega_0 t} dt + \ldots + \int_0^T c_{-2} e^{-j2\Omega_0 t} e^{-jk\Omega_0 t} dt + \int_0^T c_{-1} e^{-j\Omega_0 t} e^{-jk\Omega_0 t} dt + \int_0^T c_0 e^{-jk\Omega_0 t} dt + \int_0^T c_1 e^{j\Omega_0 t} e^{-jk\Omega_0 t} dt + \int_0^T c_2 e^{j2\Omega_0 t} e^{-jk\Omega_0 t} dt + \ldots + \int_0^T c_k\, dt + \ldots \quad \text{.....(4.11)}$$

$$= \int_0^T c_k\, dt = c_k [t]_0^T = c_k [T - 0] = T c_k$$

In equation (4.11) all definite integrals will be zero, except $\int_0^T c_k\, dt$

$$\therefore c_k = \frac{1}{T}\int_0^T x(t) e^{-jk\Omega_0 t} dt \quad \text{.....(4.12)}$$

The equation (4.12) gives the K[th] coefficient, c_k. Hence the n[th] coefficient, c_n is given by,

$$c_n = \frac{1}{T}\int_0^T x(t) e^{-jn\Omega_0 t} dt$$

4.3.4 Relation Between Fourier Coefficients of Trigonometric and Exponential Form

The relation between Fourier coefficients of trigonometric form and exponential form are given below.

$$c_0 = \frac{a_0}{2} \quad \text{.....(4.13)}$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad \text{for } n = 1, 2, 3, 4, \ldots \quad \text{.....(4.14)}$$

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$$c_{-n} = \frac{1}{2}(a_n + jb_n) \quad \text{for } -n = -1, -2, -3, -4, \ldots\ldots \qquad \ldots..(4.15)$$

$$\therefore |c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \quad \text{for all values of n, except when } n = 0. \qquad \ldots..(4.16)$$

Proof :

Consider the trigonometric form of Fourier series of x(t), (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$\cos\theta =$	$\dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\sin\theta =$	$\dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\Omega_0 t} + e^{-jn\Omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\Omega_0 t} - e^{-jn\Omega_0 t}}{2j}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} e^{jn\Omega_0 t} + \frac{a_n}{2} e^{-jn\Omega_0 t} - j\frac{b_n}{2} e^{jn\Omega_0 t} + j\frac{b_n}{2} e^{-jn\Omega_0 t}\right]$$

$$\frac{1}{j} = \frac{j}{j^2} = -j$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} - j\frac{b_n}{2}\right] e^{jn\Omega_0 t} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} + j\frac{b_n}{2}\right] e^{-jn\Omega_0 t}$$

Let, $c_0 = \dfrac{a_0}{2}$; $c_n = \dfrac{a_n - jb_n}{2}$; $c_n^* = \dfrac{a_n + jb_n}{2}$

$$\therefore x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t} + \sum_{n=1}^{\infty} c_n^* e^{-jn\Omega_0 t}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t} + \sum_{n=-\infty}^{-1} c_{-n} e^{jn\Omega_0 t}$$

$$C_{-n} = C_n^*$$

$$= \sum_{n=-\infty}^{-1} c_{-n} e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t}$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

$e^{jn\Omega_0 t} = 1$ for $n = 0$

$$\therefore c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad \text{for } n = 1, 2, 3, \ldots\ldots, \infty$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) \quad \text{for } -n = -1, -2, -3, \ldots\ldots, -\infty$$

4.3.5 Frequency Spectrum(or Line Spectrum) of Periodic Continuous Time Signals

Let x(t) be a periodic continuous time signal. Now, exponential form of Fourier series of x(t) is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

where, c_n is the Fourier coefficient of n^{th} harmonic component.

The Fourier coefficient, c_n is a complex quantity and so it can be expressed in the polar form as shown below.

$$c_n = |c_n| \angle c_n$$

where, $|c_n|$ = Magnitude of c_n ; $\angle c_n$ = Phase of c_n

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4.4 Fourier Coefficients of Signals With Symmetry

4.4.1 Even Symmetry

A signal, x(t) is called ***even signal***, if the signal satisfies the condition $x(-t) = x(t)$.

The waveform of an even periodic signal exhibits symmetry with respect to t = 0 (i.e., with respect to vertical axis) and so the symmetry of a waveform with respect to t = 0 or vertical axis is called ***even symmetry***.

> Examples of even signals are,
> $x(t) = 1 + t^2 + t^4 + t^6$
> $x(t) = A \cos \Omega_0 t$

In order to determine the even symmetry of a waveform, fold the waveform with respect to vertical axis. After folding, if the waveshape remains same then it is said to have even symmetry.

For even signals the Fourier coefficient a_0 is optional, a_n exists and b_n are zero. The Fourier coefficient a_0 is zero if the average value of one period is equal to zero. For an even signal the Fourier coefficients are given by,

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\cos n\Omega_0 t\,dt \;; \quad b_n = 0$$

Proof :

Consider the equation for a_0, (equation (4.2)).

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{T}\int_0^{T/2} x(-t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$= \frac{2}{T}\int_0^{T/2} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2} x(t)\,dt$$

> Dividing the integral into two parts.

> Change of integral index,
> Let, $t = -\tau$; $\therefore dt = -d\tau$
> When $t = 0$, $\tau = -t = 0$
> When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

> Since τ is dummy variable, Let $\tau = t$.

> Since x(t) is even, $x(-t) = x(t)$

Consider the equation for a_n, (equation (4.3)).

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\cos n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\cos n\Omega_0 \tau\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^{T/2} x(-t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

> Dividing the integral into two parts.

> Change of integral index,
> Let, $t = -\tau$; $\therefore dt = -d\tau$
> When $t = 0$, $\tau = -t = 0$
> When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

> $\cos(-\theta) = \cos\theta$

> Since τ is dummy variable, Let $\tau = t$.

> Since x(t) is even, $x(-t) = x(t)$

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Consider the equation for b_n, (equation (4.4)).

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\Omega_0 t\, dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\sin n\Omega_0 t\, dt + \frac{2}{T}\int_{0}^{T/2} x(t)\sin n\Omega_0 t\, dt$$

Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\sin n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_{0}^{T/2} x(t)\sin n\Omega_0 t\, dt$$

Change of integral index, Let, $t = -\tau$; $\therefore dt = -d\tau$
When $t = 0$, $\tau = -t = 0$
When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= -\frac{2}{T}\int_{0}^{T/2} x(-\tau)\sin n\Omega_0 \tau\, d\tau + \frac{2}{T}\int_{0}^{T/2} x(t)\sin n\Omega_0 t\, dt$$

$\sin(-\theta) = -\sin\theta$

$$= -\frac{2}{T}\int_{0}^{T/2} x(-t)\sin n\Omega_0 t\, dt + \frac{2}{T}\int_{0}^{T/2} x(t)\sin n\Omega_0 t\, dt$$

Since τ is dummy variable, Let $\tau = t$

$$= -\frac{2}{T}\int_{0}^{T/2} x(t)\sin n\Omega_0 t\, dt + \frac{2}{T}\int_{0}^{T/2} x(t)\sin n\Omega_0 t\, dt = 0$$

Since $x(t)$ is even, $x(-t) = x(t)$

The waveform of some even periodic signals and their Fourier series are given below.

The waveform shown in fig 4.4, has even symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $b_n = 0$ and a_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of cosine terms. The trigonometric Fourier series representation of the waveform of fig 4.4 is given by equation (4.17). [Please refer example 4.1 for the derivation of Fourier series]

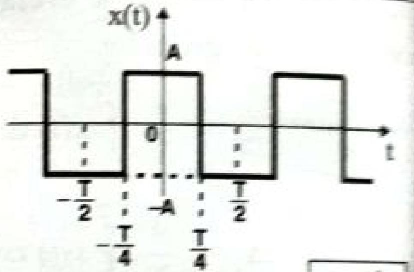

Fig 4.4.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{4A}{\pi}\left[\frac{\cos\Omega_0 t}{1} - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \frac{\cos 9\Omega_0 t}{9} - \ldots\right] \quad \ldots(4.17)$$

The waveform shown in fig 4.5, has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is substracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The trigonometric Fourier series representation of the waveform of fig 4.5 is given by equation (4.18). [Please refer example 4.3 for the derivation of Fourier series]

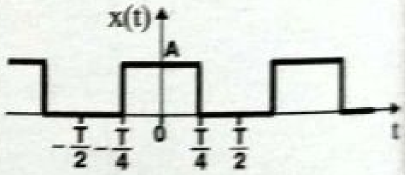

Fig 4.5.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\frac{\cos\Omega_0 t}{1} - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \frac{\cos 9\Omega_0 t}{9} - \ldots\right] \ldots(4.18)$$

The waveform shown in fig 4.6 has even symmetry and so $b_n = 0$. The trigonometric Fourier series representation of the waveform of fig 4.6 is given by equation (4.19).

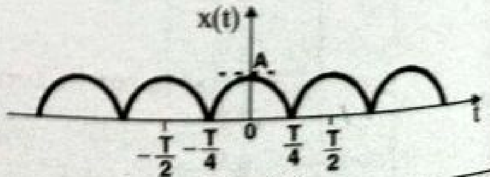

Fig 4.6.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{2A}{\pi} + \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{(2^2 - 1)} - \frac{\cos 4\Omega_0 t}{(4^2 - 1)} + \frac{\cos 6\Omega_0 t}{(6^2 - 1)} - \frac{\cos 8\Omega_0 t}{(8^2 - 1)} + \ldots\right] \quad \ldots(4.19)$$

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The waveform shown in fig 4.7 has even symmetry and so $b_n = 0$. The trigonometric Fourier series representation of the waveform of fig 4.7 is given by equation (4.20). [Please refer example 4.4 for the derivation of Fourier series].

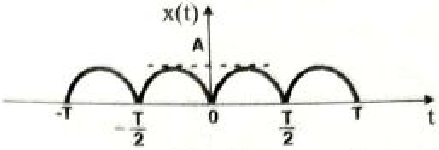

Fig 4.7.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{(2^2 - 1)} + \frac{\cos 4\Omega_0 t}{(4^2 - 1)} + \frac{\cos 6\Omega_0 t}{(6^2 - 1)} + \frac{\cos 8\Omega_0 t}{(8^2 - 1)} + ...\right] \quad .....(4.20)$$

The waveform shown in fig. 4.8 has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is subtracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The Fourier series representation of the waveform of fig 4.8 is given by equation (4.21). [Please refer example 4.2 for the derivation of Fourier series].

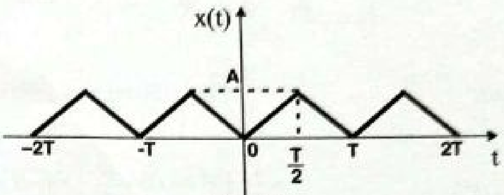

Fig 4.8.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \frac{\cos 7\Omega_0 t}{7^2} + ...\right] \quad .....(4.21)$$

The waveform shown in fig. 4.9 has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is subtracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The Fourier series representation of the waveform of fig 4.9 is given by equation (4.22). [Please refer example 4.11 for the derivation of Fourier series].

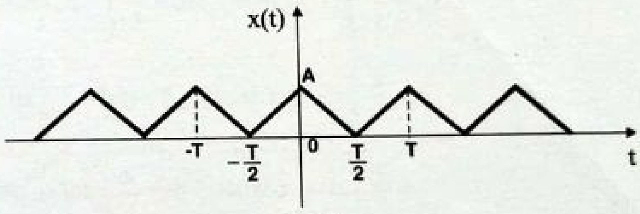

Fig 4.9.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} + \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \frac{\cos 7\Omega_0 t}{7^2} + ...\right] \quad .....(4.22)$$

4.4.2 Odd Symmetry

A signal, x(t) is called ***odd signal*** if it satisfies the condition ***x(−t) = −x(t)***.

The waveform of odd periodic signal will exhibit anti-symmetry with respect to t = 0 (i.e., with respect to vertical axis) and so the anti-symmetry of a waveform with respect to t = 0 or vertical axis is called ***odd symmetry***.

> Examples of odd signals are,
>
> $x(t) = t + t^3 + t^5 + t^7$
>
> $x(t) = A \sin \Omega_0 t$

In order to determine the odd symmetry of a waveform, invert either the right side (or the left side) of the waveform with respect to horizontal axis and then fold the waveform with respect to vertical axis. After inverting one half and folding, if the waveshape remains same then it is said to have odd symmetry.

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For odd signals a_0 and a_n are zero and b_n exists. For odd signal the Fourier coefficients are given by,

$$a_0 = 0 \quad : \quad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\,\sin n\Omega_0 t\,dt \quad \text{or} \quad b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,\sin n\Omega_0 t\,dt$$

Proof :

Consider the equation for a_0, (equation (4.2)).

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

> Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

> Change of integral index, Let, $t = -\tau$; $\therefore dt = -d\tau$
> When $t = 0$, $\tau = -t = 0$
> When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{T}\int_0^{T/2} x(-t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

> Since τ is dummy variable, Let $\tau = t$.

$$= -\frac{2}{T}\int_0^{T/2} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt = 0$$

> Since x(t) is odd, $x(-t) = -x(t)$

Consider the equation for a_n, (equation (4.3)).

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

> Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\cos n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

> Change of integral index, Let, $t = -\tau$; $\therefore dt = -d\tau$
> When $t = 0$, $\tau = -t = 0$
> When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\cos n\Omega_0 \tau\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

> $\cos(-\theta) = \cos\theta$

$$= \frac{2}{T}\int_0^{T/2} x(-t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

> Since τ is dummy variable, Let $\tau = t$.

$$= -\frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = 0$$

> Since x(t) is odd, $x(-t) = -x(t)$

Consider the equation for b_n, (equation (4.4)).

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\sin n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

> Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\sin n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

> Change of integral index, Let, $t = -\tau$; $\therefore dt = -d\tau$
> When $t = 0$, $\tau = -t = 0$
> When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= -\frac{2}{T}\int_0^{T/2} x(-\tau)\sin n\Omega_0 \tau\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

> $\sin(-\theta) = -\sin\theta$

$$= -\frac{2}{T}\int_0^{T/2} x(-t)\sin n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

> Since τ is dummy variable, Let $\tau = t$.

$$= \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

> Since x(t) is odd, $x(-t) = -x(t)$

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The waveform of some odd periodic signals and their Fourier series are given below. Certain signals will become odd after subtraction of the dc component ($a_0/2$), such a signal waveform is shown in fig 4.13.

The waveform shown in fig 4.10 has odd symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.10 is given by equation (4.23). [Please refer example 4.5 for derivation of Fourier series].

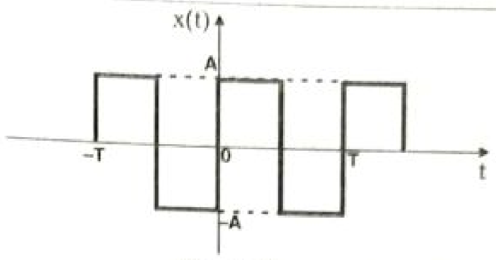

Fig 4.10.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{4A}{\pi}\left[\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \frac{\sin 7\Omega_0 t}{7} + ...\right] \quad .....(4.23)$$

The waveform shown in fig 4.11 has odd symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.11 is given by equation (4.24). [Please refer example 4.6 for derivation of Fourier series].

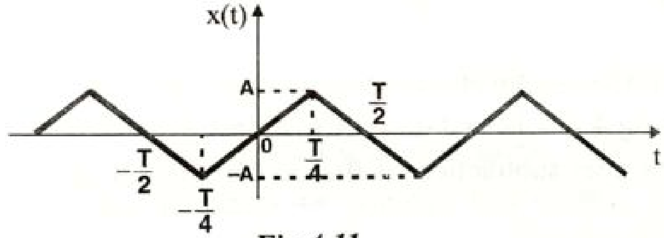

Fig 4.11.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{8A}{\pi^2}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 3\Omega_0 t}{3^2} + \frac{\sin 5\Omega_0 t}{5^2} - \frac{\sin 7\Omega_0 t}{7^2} + ...\right] \quad .....(4.24)$$

The waveform shown in fig 4.12 has odd symmetry and so $a_0 = 0$, $a_n = 0$, and b_n exists for all values of n. Hence the Fourier series has both even and odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.12 is given by equation (4.25). [Please refer example 4.7 for derivation of Fourier series].

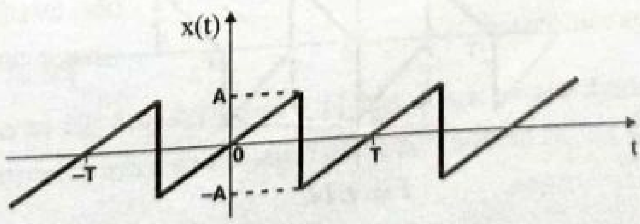

Fig 4.12.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{2A}{\pi}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} - \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} - .....\right] \quad .....(4.25)$$

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The waveform shown in fig 4.13 is neither even nor odd. But it can be shown that if the dc component ($a_0/2$) is subtracted from this waveform it becomes odd signal. Hence the Fourier coefficients $a_n = 0$ and b_n exists for all values of n. Therefore the Fourier series has a dc component and all harmonics (both even and odd harmonics) of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.13 is given by equation (4.26). [Please refer example 4.8 for derivation of Fourier series].

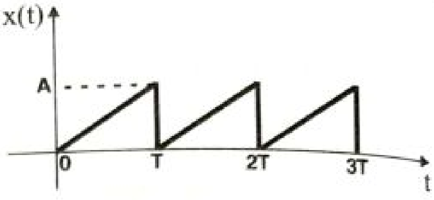

Fig 4.13.

$$\Omega_0 = \frac{2\pi}{T}$$

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} + \ldots\right] \quad .....(4.26)$$

4.4.3 Half Wave Symmetry (or Alternation Symmetry)

The periodic waveforms in which each period/cycle consists of two equal and opposite half period/cycle are called alternating waveforms, because this type of waveform will have alternate positive and negative half cycles. Such waveforms are said to have ***half wave symmetry*** or ***alternation symmetry***.

The waveforms with half wave symmetry will satisfy the condition,

$$x\left(t \pm \frac{T}{2}\right) = -x(t)$$

When a waveform has half wave symmetry, the Fourier series will consist of odd harmonic terms alone. The waveforms shown in fig 4.4, 4.10 and 4.11 exhibit half wave symmetry. Certain waveform will exhibit half wave symmetry after subtraction of the dc component ($a_0/2$), such waveforms are shown in fig 4.5, 4.8 and 4.9.

Some of the waveforms with only half wave symmetry and their Fourier series are given below.

The waveform shown in fig 4.14 has half wave symmetry. Hence the Fourier series consists of odd harmonic terms alone. The trigonometric Fourier series representation of the waveform of fig 4.14 is given by equation (4.27). [Please refer example 4.10 for the derivation of Fourier series].

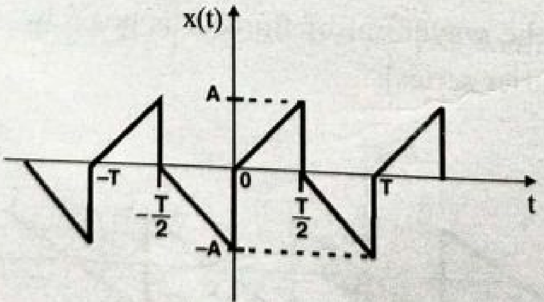

Fig 4.14.

$$\Omega_0 = \frac{2\pi}{T}$$

$$x(t) = -\frac{4A}{\pi^2}\left(\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \ldots\right) + \frac{2A}{\pi}\left(\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \ldots\right) \quad .....(4.27)$$

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The waveform shown in fig 4.15 has half wave symmemtry. Hence the Fourier series consists of odd harmonic terms alone. The trigonometric Fourier series representation of the waveform of fig 4.15 is given by equation (4.28).

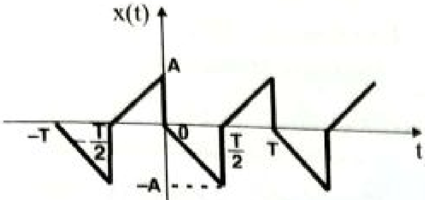

Fig 4.15.

$$x(t) = \frac{4A}{\pi^2}\left(\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right) - \frac{2A}{\pi}\left(\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + ...\right) \quad \boxed{\Omega_0 = \frac{2\pi}{T}} \quad .....(4.28)$$

4.4.4 Quarter Wave Symmetry

A waveform with half wave symmetry if in addition has even/odd symmetry then it is said to have ***quarter wave symmetry***. In a waveform with quarter wave symmetry, each quarter period will have identical shape, but may have opposite sign. The existence of the type of Fourier coefficients for waveform with quarter wave symmetry is shown below.

$$x\left(t \pm \frac{T}{2}\right) = -x(t)$$

x(t) has half wave symmerty.

Fourier series has odd harmonic terms.

↓

$x(-t) = x(t)$ and $x\left(t \pm \frac{T}{2}\right) = -x(t)$

x(t) has even and half wave symmetries. (i.e., x(t) has quarter wave symmetry). Fourier series will have odd harmonics of cosine terms.

$x(-t) = -x(t)$ and $x\left(t \pm \frac{T}{2}\right) = -x(t)$

x(t) has odd and half wave symmetries. (i.e., x(t) has quarter wave symmetry). Fourier series will have odd harmonics of sine terms.

The waveforms shown in fig 4.4, 4.10 and 4.11 has quarter wave symmetry. Certain waveforms will exhibit quarter wave symmetry after subtraction of dc component ($a_0/2$), such waveforms are shown in fig 4.5, 4.8 and 4.9.

4.5 Properties of Fourier Series

The properties of exponential form of Fourier series coefficients are listed in table 4.1. The proof of these properties are left as exercise to the readers.

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Table 4.1 : Properties of Exponential Form of Fourier Series Coefficients

Note : c_n and d_n are exponential form of Fourier series coefficients of x(t) and y(t) respectively.

Property	Continuous time periodic signal	Fourier series coefficients
Linearity	$A\,x(t) + B\,y(t)$	$A\,c_n + B\,d_n$
Time shifting	$x(t - t_0)$	$c_n\, e^{-jn\Omega_0 t_0}$
Frequency shifting	$e^{-jm\Omega_0 t}\, x(t)$	c_{n-m}
Conjugation	$x^*(t)$	c^*_{-n}
Time reversal	$x(-t)$	c_{-n}
Time scaling	$x(\alpha t)\ ;\ \alpha > 0$ (x(t) is period with period T/α)	c_n (No change in Fourier coefficient)
Multiplication	$x(t)\ y(t)$	$\sum_{m=-\infty}^{+\infty} c_m\, d_{n-m}$
Differentiation	$\frac{d}{dt}x(t)$	$jn\Omega_0\, c_n$
Integration	$\int_{-\infty}^{t} x(t)\, dt$ (Finite valued and periodic only if $a_0 = 0$)	$\frac{i}{jn\Omega_0}\, c_n$
Periodic convolution	$\int_T x(\tau)\, y(t-\tau)\, d\tau$	$T\, c_n\, d_n$
Symmetry of real signals	x(t) is real	$c_n = c^*_{-n}$ $\lvert c_n\rvert = \lvert c_{-n}\rvert\ ;\ \angle c_n = -\angle c_{-n}$ $Re\{c_n\} = Re\{c_{-n}\}$ $Im\{c_n\} = -Im\{c_{-n}\}$
Real and even	x(t) is real and even	c_n are real and even
Real and odd	x(t) is real and odd	c_n are imaginary and odd
Parseval's relation	Average power, P of x(t) is defined as, $P = \frac{1}{T}\int_T \lvert x(t)\rvert^2\, dt$	The average power, P in terms of Fourier series coefficients is, $P = \sum_{n=-\infty}^{+\infty} \lvert c_n\rvert^2$
	$\frac{1}{T}\int_T \lvert x(t)\rvert^2\, dt = \sum_{n=-\infty}^{+\infty} \lvert c_n\rvert^2$	

Note : 1. The term $\lvert c_n\rvert^2$ respresent the power in n^{th} harmonic component of x(t). The total average power in a periodic signal is equal to the sum of power in all of its harmonics.

2. The term $\lvert c_n\rvert^2$ for n = 0, 1, 2, ... is the distribution of power as a function of frequency and so it is called ***power density spectrum*** *or* ***power spectral density*** *of the periodic signal*

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4.8 Solved Problems in Fourier Series

Example 4.1

Determine the trigonometric form of Fourier series of the waveform shown in fig 4.1.1.

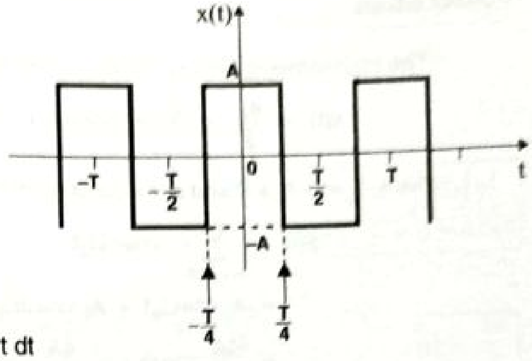

Fig 4.1.1.

Solution

The waveform shown in fig 4.1.1 has even symmetry, half wave symmetry and quarter wave symmetry.

$$\therefore\ a_0 = 0,\quad b_n = 0 \quad \text{and} \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of the square wave is,

$$x(t) = A \quad ; \quad \text{for} \quad t = 0 \text{ to } \frac{T}{4}$$

$$= -A \quad ; \quad \text{for} \quad t = \frac{T}{4} \text{ to } \frac{T}{2}$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/4} A\cos n\Omega_0 t\,dt + \frac{4}{T}\int_{T/4}^{T/2} (-A)\cos n\Omega_0 t\,dt$$

$$= \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^{T/4} - \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/4}^{T/2} = \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/4} - \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_{T/4}^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right] - \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{2}\right)}{n\frac{2\pi}{T}} - \frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}}\right]$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi}\sin\frac{n\pi}{2} - 0\right] - \frac{4A}{T}\left[\frac{T}{2n\pi}\sin n\pi - \frac{T}{2n\pi}\sin\frac{n\pi}{2}\right] \qquad \boxed{\sin 0 = 0} \qquad \boxed{\sin n\pi = 0 \text{ for integer } n}$$

$$= \frac{2A}{n\pi}\sin\frac{n\pi}{2} + \frac{2A}{n\pi}\sin\frac{n\pi}{2} = \frac{4A}{n\pi}\sin\frac{n\pi}{2}$$

For even values of n, $\sin\frac{n\pi}{2} = 0$

For odd values of n, $\sin\frac{n\pi}{2} = \pm 1$

$$\therefore\ a_n = 0 \quad ; \text{ for even values of } n$$

$$a_n = \frac{4A}{n\pi}\sin\frac{n\pi}{2} \quad ; \text{ for odd values of } n$$

$$\therefore\ a_1 = \frac{4A}{1\times\pi}\sin\frac{\pi}{2} = +\frac{4A}{\pi}$$

$$a_3 = \frac{4A}{3\times\pi}\sin\frac{3\pi}{2} = -\frac{4A}{3\pi}$$

$$a_5 = \frac{4A}{5\times\pi}\sin\frac{5\pi}{2} = +\frac{4A}{5\pi}$$

$$a_7 = \frac{4A}{7\times\pi}\sin\frac{7\pi}{2} = -\frac{4A}{7\pi} \quad \text{and so on.}$$

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Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $b_n = 0$ and a_n exists only for odd values of n.

$$\therefore x(t) = \sum_{n=\text{odd}} a_n \cos n\Omega_0 t$$

$$= a_1 \cos\Omega_0 t + a_3 \cos 3\Omega_0 t + a_5 \cos 5\Omega_0 t + a_7 \cos 7\Omega_0 t + ...$$

$$= \frac{4A}{\pi}\cos\Omega_0 t - \frac{4A}{3\pi}\cos 3\Omega_0 t + \frac{4A}{5\pi}\cos 5\Omega_0 t - \frac{4A}{7\pi}\cos 7\Omega_0 t + ...$$

$$= \frac{4A}{\pi}\left[\cos\Omega_0 t - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + ...\right]$$

Example 4.2

Find the Fourier series of the waveform shown in fig 4.2.1.

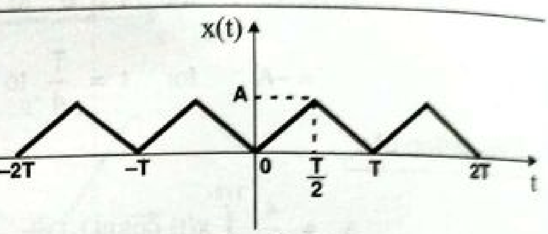

Fig 4.2.1.

Solution

The given waveform has even symmetry and so $b_n = 0$

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \quad ; \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt \quad ; \quad b_n = 0$$

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\frac{T}{2}, A\right]$

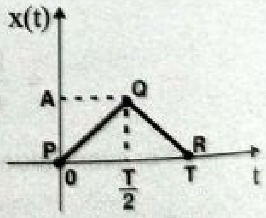

Fig 1.

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{-2t}{T} \Rightarrow x(t) = \frac{2A}{T}t$$

$$\therefore x(t) = \frac{2A}{T}t \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2}\frac{2A}{T}t\,dt = \frac{8A}{T^2}\int_0^{T/2} t\,dt$$

$$= \frac{8A}{T^2}\left[\frac{t^2}{2}\right]_0^{T/2} = \frac{8A}{T^2}\left[\frac{T^2}{8} - 0\right] = A$$

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Evaluation a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2}\frac{2A}{T}t\cos n\Omega_0 t\,dt = \frac{8A}{T^2}\int_0^{T/2} t\cos n\Omega_0 t\,dt$$

$$= \frac{8A}{T^2}\left[t\frac{\sin n\Omega_0 t}{n\Omega_0} - \int 1\times\left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^{T/2}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \cos\Omega_0 t$

$$= \frac{8A}{T^2}\left[\frac{t\sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} = \frac{8A}{T^2}\left[\frac{t\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^{T/2}$$

$$\Omega_0 = \frac{2\pi}{T}$$

$$= \frac{8A}{T^2}\left[\frac{T}{2}\frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} - \frac{0\times\sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\sin 0 = 0$
$\cos 0 = 1$

$$= \frac{8A}{T^2}\left[\frac{T^2}{4n\pi}\sin n\pi + \frac{T^2}{4n^2\pi^2}\cos n\pi - \frac{T^2}{4n^2\pi^2}\right] = \frac{2A}{n^2\pi^2}[\cos n\pi - 1]$$

$\sin n\pi = 0$ for integer values of n

For even integer values of n, $\cos n\pi = +1$

For odd integer values of n, $\cos n\pi = -1$

$\therefore\ a_n = 0$; for even values of n, and

$$a_n = \frac{2A}{n^2\pi^2}[\cos n\pi - 1] = -\frac{4A}{n^2\pi^2} \quad ; \text{ for odd values of n.}$$

$$\therefore\ a_1 = -\frac{4A}{1^2\pi^2} \quad ; \quad a_3 = -\frac{4A}{3^2\pi^2} \quad ; \quad a_5 = -\frac{4A}{5^2\pi^2} \quad ; \text{ and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n\sin n\Omega_0 t$$

Here, $b_n = 0$, and a_n exists only for odd values of n.

$$\therefore\ x(t) = \frac{a_0}{2} + \sum_{n=\text{odd}} a_n\cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5\cos 5\Omega_0 t + ...$$

$$= \frac{A}{2} - \frac{4A}{1^2\pi^2}\cos\Omega_0 t - \frac{4A}{3^2\pi^2}\cos 3\Omega_0 t - \frac{4A}{5^2\pi^2}\cos 5\Omega_0 t - ......$$

$$= \frac{A}{2} - \frac{4A}{\pi^2}\left[\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right]$$

Example 4.3

Determine the trigonometric form of Fourier series of the waveform shown in fig 4.3.1.

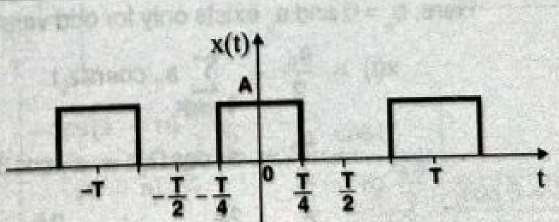

Fig 4.3.1.

Solution

The waveform of fig 4.3.1 has even symmetry.

$$\therefore\ b_n = 0, \quad a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \quad ; \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

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The mathematical equation of the given periodic rectangular pulse is,

$$x(t) = A \;;\quad \text{for } t = 0 \text{ to } \frac{T}{4}$$

$$= 0 \;;\quad \text{for } t = \frac{T}{4} \text{ to } \frac{T}{2}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/4} A\,dt = \frac{4}{T}\left[At\right]_0^{T/4}$$

$$= \frac{4}{T}\left[A\frac{T}{4} - 0\right] = A$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/4} A\cos n\Omega_0 t\,dt = \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^{T/4}$$

$$= \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/4} = \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}\frac{T}{4}}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right]$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}} \quad \boxed{\sin 0 = 0}$$

$$= \frac{4A}{T} \times \frac{T}{2n\pi}\sin\frac{n\pi}{2} = \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

For even values of n, $\sin\frac{n\pi}{2} = 0$

For odd values of n, $\sin\frac{n\pi}{2} = \pm 1$

$\therefore\ a_n = 0$; for even values of n, and

$$a_n = \frac{2A}{n\pi}\sin\frac{n\pi}{2} \;;\quad \text{for odd values of n.}$$

$$\therefore\ a_1 = \frac{2A}{1\times\pi}\sin\frac{\pi}{2} = +\frac{2A}{\pi}$$

$$a_3 = \frac{2A}{3\times\pi}\sin\frac{3\pi}{2} = -\frac{2A}{3\pi}$$

$$a_5 = \frac{2A}{5\times\pi}\sin\frac{5\pi}{2} = +\frac{2A}{5\pi}$$

$$a_7 = \frac{2A}{7\times\pi}\sin\frac{7\pi}{2} = -\frac{2A}{7\pi} \quad \text{and so on.}$$

Fourier Series

The trigonomteric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$ and a_n exists only for odd values of n.

$$\therefore\ x(t) = \frac{a_0}{2} + \sum_{n=\text{odd}} a_n \cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5\cos 5\Omega_0 t + a_7\cos 7\Omega_0 t + \ldots..$$

$$= \frac{A}{2} + \frac{2A}{\pi}\cos\Omega_0 t - \frac{2A}{3\pi}\cos 3\Omega_0 t + \frac{2A}{5\pi}\cos 5\Omega_0 t - \frac{2A}{7\pi}\cos 7\Omega_0 t + \ldots..$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left[\cos\Omega_0 t - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \ldots..\right]$$

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Example 4.4

Determine the trigonometric form of Fourier series of the full wave rectified sine wave shown in fig 4.4.1.

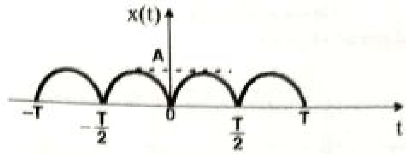

Fig 4.4.1.

Solution

The waveform shown in fig 4.4.1 is the output of full wave rectifier and it has even symmetry.

$$\therefore\ b_n = 0,\quad a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt\ ;\quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of full wave rectified output is,

$$x(t) = A\sin\Omega_0 t\ ;\quad \text{for } t = 0 \text{ to } \frac{T}{2} \quad \text{and} \quad \Omega_0 = \frac{2\pi}{T}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\,dt = \frac{4A}{T}\left[-\frac{\cos\Omega_0 t}{\Omega_0}\right]_0^{T/2}$$

$$= \frac{4A}{T}\left[-\frac{\cos\frac{2\pi}{T}t}{\frac{2\pi}{T}}\right]_0^{T/2} = \frac{4A}{T}\left[-\frac{\cos\frac{2\pi}{T}\frac{T}{2}}{\frac{2\pi}{T}} + \frac{\cos 0}{\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2A}{\pi}\left[1+1\right] = \frac{4A}{\pi} \qquad \boxed{\cos\pi = -1}\ \boxed{\cos 0 = 1}$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\cos n\Omega_0 t\,dt$$

$$= \frac{4A}{T}\int_0^{T/2}\frac{\sin(\Omega_0 t + n\Omega_0 t) + \sin(\Omega_0 t - n\Omega_0 t)}{2}\,dt \qquad \boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{2A}{T}\int_0^{T/2}\sin(1+n)\Omega_0 t\,dt + \frac{2A}{T}\int_0^{T/2}\sin(1-n)\Omega_0 t\,dt$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\Omega_0 t}{(1+n)\Omega_0}\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos(1-n)\Omega_0 t}{(1-n)\Omega_0}\right]_0^{T/2}$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}}\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}}\right]_0^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} + \frac{\cos 0}{(1+n)\frac{2\pi}{T}}\right] + \frac{2A}{T}\left[\frac{-\cos(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} + \frac{\cos 0}{(1-n)\frac{2\pi}{T}}\right] \qquad \boxed{\cos 0 = 1}$$

$$= -\frac{A\cos(1+n)\pi}{(1+n)\pi} + \frac{A}{(1+n)\pi} - \frac{A\cos(1-n)\pi}{(1-n)\pi} + \frac{A}{(1-n)\pi} \qquad .....(1)$$

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The equation (1) for a_n can be evaluated for all values of n except n = 1. For n = 1, a_n has to be estimated seperately as shown below.

$$a_1 = \frac{4}{T}\int_0^{T/2} x(t)\cos\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\cos\Omega_0 t\,dt$$

$$= \frac{4}{T}\int_0^{T/2} A\frac{\sin 2\Omega_0 t}{2}\,dt = \frac{2A}{T}\int_0^{T/2}\sin 2\Omega_0 t\,dt$$

$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$$= \frac{2A}{T}\left[\frac{-\cos 2\Omega_0 t}{2\Omega_0}\right]_0^{T/2} = \frac{2A}{T}\left[\frac{-\cos\left(2\times\frac{2\pi}{T}\times\frac{T}{2}\right)}{2\times\frac{2\pi}{T}} + \frac{\cos 0}{2\times\frac{2\pi}{T}}\right]$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$\boxed{\cos 2\pi = \cos 0 = 1}$

$$= \frac{2A}{T}\left[-\frac{T}{4\pi}\cos 2\pi + \frac{T}{4\pi}\right] = \frac{2A}{T}\left[-\frac{T}{4\pi} + \frac{T}{4\pi}\right] = 0$$

For values of n > 1, the a_n are calculated using equation (1) as shown below.

$$\therefore a_n = -\frac{A\cos(1+n)\pi}{(1+n)\pi} + \frac{A}{(1+n)\pi} - \frac{A\cos(1-n)\pi}{(1-n)\pi} + \frac{A}{(1-n)\pi}$$

When n is even integer, (1 + n) and (1 – n) will be odd, $\therefore \cos(1+n)\pi = -1$; $\cos(1-n)\pi = -1$

When n is odd integer, (1 + n) and (1 – n) will be even, $\therefore \cos(1+n)\pi = 1$; $\cos(1-n)\pi = 1$

$\therefore a_n = 0$; for odd values of n

$$a_n = \frac{A}{(1+n)\pi} + \frac{A}{(1+n)\pi} + \frac{A}{(1-n)\pi} + \frac{A}{(1-n)\pi} \quad ; \text{ for even values of n}$$

$$= \frac{2A}{(1+n)\pi} + \frac{2A}{(1-n)\pi} = \frac{2A(1-n) + 2A(1+n)}{(1+n)(1-n)\pi} = \frac{4A}{(1-n^2)\pi}$$

$$\therefore a_2 = \frac{4A}{(1-2^2)\pi} = -\frac{4A}{3\pi}$$

$$a_4 = \frac{4A}{(1-4^2)\pi} = -\frac{4A}{15\pi}$$

$$a_6 = \frac{4A}{(1-6^2)\pi} = -\frac{4A}{35\pi}$$

$$a_8 = \frac{4A}{(1-8^2)\pi} = -\frac{4A}{63\pi} \quad \text{and so on}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n\sin n\Omega_0 t$$

Here, $b_n = 0$, and a_n exists only for even values of n.

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=\text{even}} a_n\cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_2\cos 2\Omega_0 t + a_4\cos 4\Omega_0 t + a_6\cos 6\Omega_0 t + a_8\cos 8\Omega_0 t + ...$$

$$= \frac{2A}{\pi} - \frac{4A}{3\pi}\cos 2\Omega_0 t - \frac{4A}{15\pi}\cos 4\Omega_0 t - \frac{4A}{35\pi}\cos 6\Omega_0 t - \frac{4A}{63\pi}\cos 8\Omega_0 t - .....$$

$$= \frac{2A}{\pi} - \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{3} + \frac{\cos 4\Omega_0 t}{15} + \frac{\cos 6\Omega_0 t}{35} + \frac{\cos 8\Omega_0 t}{63} + ...\right]$$

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Example 4.5

Determine the Fourier series of the square wave shown in fig 4.5.1.

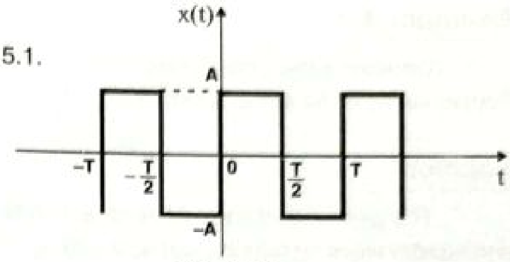

Fig 4.5.1.

Solution

The given waveform has odd symmetry, half-wave symmetry and quarter wave symmetry.

$$\therefore\ a_0 = 0,\quad a_n = 0,\quad b_n = \frac{4}{T}\int_0^{T/2} x(t)\ \sin n\Omega_0 t\ dt$$

The mathematical equation of the given waveform is,

$$x(t) = A\ ;\ \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= -A\ ;\ \text{for } t = \frac{T}{2} \text{ to } T$$

Evaluation of b_n

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\ \sin n\Omega_0 t\ dt = \frac{4}{T}\int_0^{T/2} A\ \sin n\Omega_0 t\ dt = \frac{4A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_0^{T/2}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}} \quad \boxed{\cos 0 = 1}$$

$$= \frac{4A}{T}\left[\frac{-\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/2} = \frac{4A}{T}\left[\frac{-\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos 0}{n\frac{2\pi}{T}}\right] = \frac{4A}{T}\left[-\frac{T}{2n\pi}\cos n\pi + \frac{T}{2n\pi}\right]$$

$\cos n\pi = -1,\quad$ for $n = \text{odd}$

$\cos n\pi = +1,\quad$ for $n = \text{even}$

$$\therefore\ b_n = 0\ ;\ \text{for even values of } n$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi} + \frac{T}{2n\pi}\right] = \frac{4A}{n\pi};\ \text{for odd values of } n$$

$$\therefore\ b_1 = \frac{4A}{\pi}\ ;\ b_3 = \frac{4A}{3\pi}\ ;\ b_5 = \frac{4A}{5\pi}\ \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n.

$$\therefore\ x(t) = \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + \ldots$$

$$= \frac{4A}{\pi}\sin\Omega_0 t + \frac{4A}{3\pi}\sin 3\Omega_0 t + \frac{4A}{5\pi}\sin 5\Omega_0 t + \ldots$$

$$= \frac{4A}{\pi}\left[\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \ldots\right]$$

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Example 4.6

Determine the trigonometric form of Fourier series of the signal shown in fig 4.6.1.

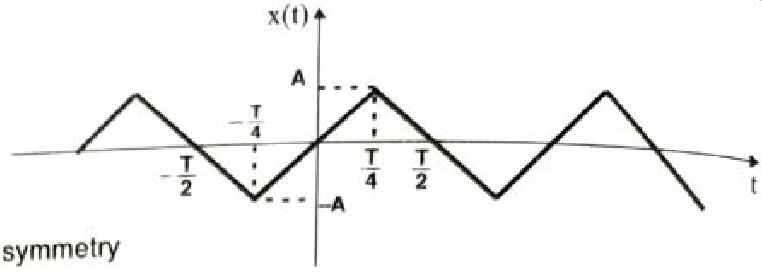

Fig 4.6.1.

Solution

The given signal has odd symmetry, half wave symmetry and quarter wave symmetry, and so $a_0 = 0$, $a_n = 0$,

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\,\sin n\Omega_0 t\,dt \quad \text{(or)} \quad b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,\sin n\Omega_0 t\,dt$$

> **Note :** *Here x(t) is governed by single mathematical equation in the range $-\frac{T}{4}$ to $+\frac{T}{4}$. And so the calculations will be simple, if the integral limit is $-\frac{T}{4}$ to $+\frac{T}{4}$*

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

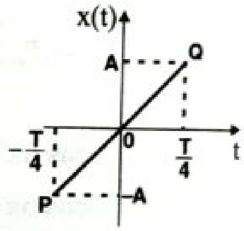

Fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = \left[-\frac{T}{4}, -A\right]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\frac{T}{4}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - (-A)}{-A - A} = \frac{t - \left(-\frac{T}{4}\right)}{-\frac{T}{4} - \frac{T}{4}} \Rightarrow \frac{x(t) + A}{-2A} = \frac{t + \frac{T}{4}}{-\frac{T}{2}}$$

$$\Downarrow$$

$$-\frac{x(t)}{2A} - \frac{1}{2} = -\frac{2t}{T} - \frac{1}{2} \Rightarrow -\frac{x(t)}{2A} = -\frac{2t}{T} \Rightarrow x(t) = \frac{4A}{T}t$$

$$\therefore x(t) = \frac{4A}{T}t \ ; \ \text{for } t = -\frac{T}{4} \text{ to } +\frac{T}{4}$$

Evaluation of b_n

$$b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,\sin n\Omega_0 t\,dt = \frac{4}{T}\int_{-T/4}^{+T/4} \frac{4A}{T}t\,\sin n\Omega_0 t\,dt = \frac{16A}{T^2}\int_{-T/4}^{+T/4} t\,\sin n\Omega_0 t\,dt$$

$$= \frac{16A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_{-T/4}^{+T/4}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{16A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_{-T/4}^{T/4} = \frac{16A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_{-T/4}^{+T/4}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{16A}{T^2}\left[-\frac{T}{4}\frac{\cos n\frac{2\pi}{T}\frac{T}{4}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{4}}{n^2\frac{4\pi^2}{T^2}} + \frac{T}{4}\frac{\cos n\frac{2\pi}{T}\left(-\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\left(-\frac{T}{4}\right)}{n^2\frac{4\pi^2}{T^2}}\right]$$

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$$b_n = \frac{16A}{T^2}\left[-\frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2} + \frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{2A}{n\pi}\cos\frac{n\pi}{2} + \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2A}{n\pi}\cos\frac{n\pi}{2} + \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

For odd integer values of n, $\sin\frac{n\pi}{2} = \pm 1$

For even integer values of n, $\sin\frac{n\pi}{2} = 0$

$$\therefore b_n = 0 \quad ; \text{ for even values of n}$$

$$= \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}; \text{ for odd values of n}$$

$$\therefore b_1 = \frac{8A}{1^2\pi^2}\sin\frac{\pi}{2} = +\frac{8A}{\pi^2}$$

$$b_3 = \frac{8A}{3^2\pi^2}\sin\frac{3\pi}{2} = -\frac{8A}{3^2\pi^2}$$

$$b_5 = \frac{8A}{5^2\pi^2}\sin\frac{5\pi}{2} = +\frac{8A}{5^2\pi^2}$$

$$b_7 = \frac{8A}{7^2\pi^2}\sin\frac{7\pi}{2} = -\frac{8A}{7^2\pi^2} \quad \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n.

$$\therefore x(t) = \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + b_7 \sin 7\Omega_0 t + \ldots$$

$$= \frac{8A}{\pi^2}\sin\Omega_0 t - \frac{8A}{3^2\pi^2}\sin 3\Omega_0 t + \frac{8A}{5^2\pi^2}\sin 5\Omega_0 t - \frac{8A}{7^2\pi^2}\sin 7\Omega_0 t + \ldots$$

$$= \frac{8A}{\pi^2}\left[\sin\Omega_0 t - \frac{\sin 3\Omega_0 t}{3^2} + \frac{\sin 5\Omega_0 t}{5^2} - \frac{\sin 7\Omega_0 t}{7^2} + \ldots\right]$$

Example 4.7

Determine the trigonometric form of Fourier series for the signal shown in fig 4.7.1.

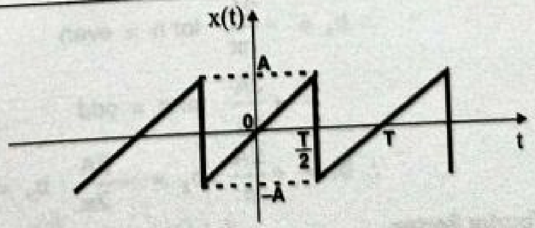

Fig 4.7.1.

Solution

The given signal has odd symmetry and so $a_0 = 0$, $a_n = 0$,

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\, dt$$

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To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

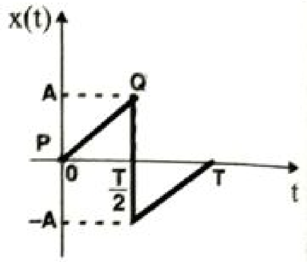

Fig 1.

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\dfrac{T}{2}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-\frac{T}{2}} \Rightarrow x(t) = \frac{2At}{T}$$

$$\therefore x(t) = \frac{2At}{T} \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of b_n

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} \frac{2At}{T}\sin n\Omega_0 t\,dt = \frac{8A}{T^2}\int_0^{T/2} t\sin n\Omega_0 t\,dt$$

$$= \frac{8A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^{T/2}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{8A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_0^{T/2} = \frac{8A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^{T/2}$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$= \frac{8A}{T^2}\left[-\frac{\frac{T}{2}\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} + \frac{0 \times \cos 0}{n\frac{2\pi}{T}} - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\boxed{\sin 0 = 0}$

$$= \frac{8A}{T^2}\left[-\frac{T^2}{4n\pi}\cos n\pi + \frac{T^2}{4n^2\pi^2}\sin n\pi\right] = -\frac{2A}{n\pi}\cos n\pi$$

$\boxed{\sin n\pi = 0 \text{ for integer } n}$

For even integer values of n, $\cos n\pi = +1$

For odd integer values of n, $\cos n\pi = -1$

$$\therefore b_n = -\frac{2A}{n\pi} \quad \text{for } n = \text{even}$$

$$= +\frac{2A}{n\pi} \quad \text{for } n = \text{odd}$$

$$\therefore b_1 = +\frac{2A}{\pi}\ ;\ b_2 = -\frac{2A}{2\pi}\ ;\ b_3 = +\frac{2A}{3\pi}\ ;\ b_4 = -\frac{2A}{4\pi}\ ;\ b_5 = +\frac{2A}{5\pi} \text{ and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

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Here, $a_0 = 0$, $a_n = 0$

$$\therefore x(t) = \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + b_4 \sin 4\Omega_0 t + b_5 \sin 5\Omega_0 t + ...$$

$$= \frac{2A}{\pi}\sin\Omega_0 t - \frac{2A}{2\pi}\sin 2\Omega_0 t + \frac{2A}{3\pi}\sin 3\Omega_0 t - \frac{2A}{4\pi}\sin 4\Omega_0 t + \frac{2A}{5\pi}\sin 5\Omega_0 t - .....$$

$$= \frac{2A}{\pi}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} - \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} - .....\right]$$

Example 4.8

Determine the trigonometric form of the Fourier series of the ramp signal shown in fig 4.8.1.

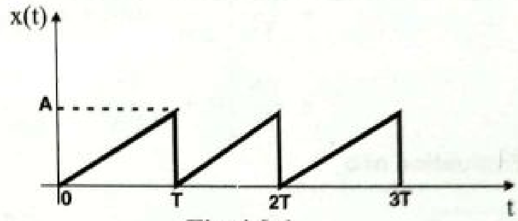

Fig 4.8.1.

Solution

The given signal is neither even nor odd.

$$\therefore a_0 = \frac{2}{T}\int_0^T x(t)\,dt \;;\; a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \;;\; b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt$$

***Note :** It can be shown that after subtracting $a_0/2$ from the signal, it becomes odd. Hence a_n will be equal to zero.*

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = [T, A]$

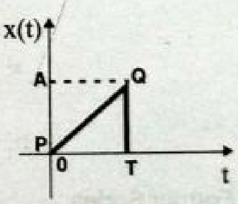

Fig 1.

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - T} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-T} \Rightarrow x(t) = \frac{At}{T}$$

$$\therefore x(t) = \frac{At}{T} \;;\; \text{for } t = 0 \text{ to } T$$

Evaluation of a_0

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_0^T \frac{A}{T}t\,dt = \frac{2A}{T^2}\int_0^T t\,dt$$

$$= \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{2A}{T^2}\left[\frac{T^2}{2} - 0\right] = A$$

Evaluation of a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_0^T \frac{At}{T}\cos n\Omega_0 t\,dt = \frac{2A}{T^2}\int_0^T t\cos n\Omega_0 t\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \cos n\Omega_0 t$

$$= \frac{2A}{T^2}\left[t\left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^T$$

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$$a_n = \frac{2A}{T^2}\left[t\frac{\sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^T = \frac{2A}{T^2}\left[\frac{t\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^T$$

$\Omega_0 = \frac{2\pi}{T}$

$$= \frac{2A}{T^2}\left[\frac{T\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} - \frac{0\times\sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\sin 0 = 0$
$\cos 0 = 1$

$$= \frac{2A}{T^2}\left[\frac{T^2}{2n\pi}\sin n2\pi + \frac{T^2}{4n^2\pi^2}\cos n2\pi - 0 - \frac{T^2}{4n^2\pi^2}\right]$$

$\sin n2\pi = 0$
$\cos n2\pi = 1$
for integer n

$$= \frac{2A}{T^2}\left[0 + \frac{T^2}{4n^2\pi^2} - \frac{T^2}{4n^2\pi^2}\right] = 0$$

Evaluation of b_n

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt = \frac{2}{T}\int_0^T \frac{At}{T}\sin n\Omega_0 t\,dt = \frac{2A}{T^2}\int_0^T t\sin n\Omega_0 t\,dt$$

$$= \frac{2A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1\times\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^T$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{2A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_0^T = \frac{2A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^T$$

$\Omega_0 = \frac{2\pi}{T}$

$$= \frac{2A}{T^2}\left[-\frac{T\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} + \frac{0\times\cos 0}{n\frac{2\pi}{T}} - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\sin 0 = 0$

$$= \frac{2A}{T^2}\left[-\frac{T^2}{2n\pi}\cos n2\pi + \frac{T^2}{4n^2\pi^2}\sin n2\pi\right] = -\frac{A}{n\pi}$$

$\sin n2\pi = 0$
$\cos n2\pi = 1$
for integer n

$\therefore b_1 = -\frac{A}{\pi}$; $b_2 = -\frac{A}{2\pi}$; $b_3 = -\frac{A}{3\pi}$; $b_4 = -\frac{A}{4\pi}$ and so on.

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_n = 0$

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + b_3\sin 3\Omega_0 t + b_4\sin 4\Omega_0 t + \,.....$$

$$= \frac{A}{2} - \frac{A}{\pi}\sin\Omega_0 t - \frac{A}{2\pi}\sin 2\Omega_0 t - \frac{A}{3\pi}\sin 3\Omega_0 t - \frac{A}{4\pi}\sin 4\Omega_0 t - \,.....$$

$$= \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 4\Omega_0 t}{4} + \,.....\right]$$

Example 4.9

Determine the Fourier series representation of the half-wave rectifier output shown in fig 4.9.1.

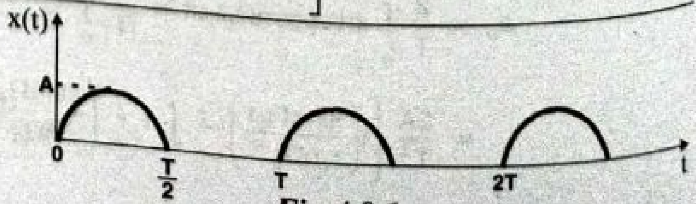

Fig 4.9.1.

Solution

The signal shown in fig 4.9.1 is neither even nor odd.

$$\therefore\ a_0 = \frac{2}{T}\int_0^T x(t)\,dt \ ;\ a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \ ;\ b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt$$

The mathematical equation representing half-wave rectified output is,

$$x(t) = A\sin\Omega_0 t \ ;\ \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= 0 \ ;\ \text{for } t = \frac{T}{2} \text{ to } T$$

Evaluation of a_0

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\,dt = \frac{2A}{T}\left[\frac{-\cos\Omega_0 t}{\Omega_0}\right]_0^{T/2}$$

$$= \frac{2A}{T}\left[\frac{-\cos\frac{2\pi}{T}t}{\frac{2\pi}{T}}\right]_0^{T/2} = \frac{2A}{T}\left[\frac{-\cos\frac{2\pi}{T}\frac{T}{2}}{\frac{2\pi}{T}} + \frac{\cos 0}{\frac{2\pi}{T}}\right]$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$= \frac{2A}{T}\left[-\frac{T}{2\pi}\cos\pi + \frac{T}{2\pi}\right] = \frac{2A}{T}\left[-\frac{T}{2\pi}\times(-1) + \frac{T}{2\pi}\right]$$

$\boxed{\cos 0 = 1,\ \cos\pi = -1}$

$$= \frac{2A}{T}\times\frac{T}{\pi} = \frac{2A}{\pi}$$

Evaluation of a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\cos n\Omega_0 t\,dt$$

$\boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$

$$= \frac{2A}{T}\int_0^{T/2}\frac{\sin(\Omega_0 t + n\Omega_0 t) + \sin(\Omega_0 t - n\Omega_0 t)}{2}\,dt = \frac{A}{T}\int_0^{T/2}\left[\sin(1+n)\Omega_0 t + \sin(1-n)\Omega_0 t\right]dt$$

$$= \frac{A}{T}\left[-\frac{\cos(1+n)\Omega_0 t}{(1+n)\Omega_0} - \frac{\cos(1-n)\Omega_0 t}{(1-n)\Omega_0}\right]_0^{T/2} = \frac{A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}} - \frac{\cos(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}}\right]_0^{T/2}$$

$$= \frac{A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} - \frac{\cos(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} + \frac{\cos 0}{(1+n)\frac{2\pi}{T}} + \frac{\cos 0}{(1-n)\frac{2\pi}{T}}\right]$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$ $\boxed{\cos 0 = 1}$

$$= \frac{A}{T}\left[-\frac{T\cos(1+n)\pi}{(1+n)2\pi} - \frac{T\cos(1-n)\pi}{(1-n)2\pi} + \frac{T}{(1+n)2\pi} + \frac{T}{(1-n)2\pi}\right]$$

The above expression for a_n can be evaluated for all values of n except for n = 1. For n = 1, a_n has to be evaluated separately as shown below.

$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$$a_1 = \frac{2}{T}\int_0^T x(t)\cos\Omega_0 t\,dt = \frac{2}{T}\int_0^T A\sin\Omega_0 t\cos\Omega_0 t\,dt$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$= \frac{2A}{T}\int_0^T\frac{\sin 2\Omega_0 t}{2}\,dt = \frac{A}{T}\left[\frac{-\cos 2\Omega_0 t}{2\Omega_0}\right]_0^T = \frac{A}{T}\left[\frac{-\cos\frac{4\pi}{T}t}{\frac{4\pi}{T}}\right]_0^T$$

$\boxed{\cos 4\pi = 1,\ \cos 0 = 1}$

$$= \frac{A}{T}\left[-\frac{T}{4\pi}\cos\frac{4\pi T}{T} + \frac{T}{4\pi}\cos 0\right] = \frac{A}{T}\left[-\frac{T}{4\pi} + \frac{T}{4\pi}\right] = 0$$

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$\therefore a_1 = 0$

$$a_n = -\frac{A\cos(1+n)\pi}{(1+n)2\pi} - \frac{A\cos(1-n)\pi}{(1-n)2\pi} + \frac{A}{(1+n)2\pi} + \frac{A}{(1-n)2\pi} \quad ; \text{ for all n except n = 1}$$

When n is even, the terms (n + 1) and (n − 1) are odd. $\therefore \cos(1+n)\pi = -1,\ \cos(1-n)\pi = -1$

When n is odd, the terms (n + 1) and (n − 1) are even. $\therefore \cos(1+n)\pi = 1,\ \cos(1-n)\pi = 1$

$\therefore a_n = 0$; for odd values of n

$$a_n = \frac{A}{(1+n)2\pi} + \frac{A}{(1-n)2\pi} + \frac{A}{(1+n)2\pi} + \frac{A}{(1-n)2\pi} \quad ; \text{ for even values of n}$$

$$= \frac{A}{(1+n)\pi} + \frac{A}{(1-n)\pi} = \frac{A(1-n) + A(1+n)}{(1+n)(1-n)\pi} = \frac{2A}{(1-n^2)\pi}$$

$$\therefore a_2 = \frac{2A}{(1-2^2)\pi} = -\frac{2A}{3\pi}$$

$$a_4 = \frac{2A}{(1-4^2)\pi} = -\frac{2A}{15\pi}$$

$$a_6 = \frac{2A}{(1-6^2)\pi} = -\frac{2A}{35\pi}$$

$$a_8 = \frac{2A}{(1-8^2)\pi} = -\frac{2A}{63\pi} \text{ and so on.}$$

Evaluation of b_n

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\,\sin n\Omega_0 t\,dt$$

$\boxed{2\sin A\sin B = \cos(A-B) - \cos(A+B)}$

$$= \frac{2A}{T}\int_0^{T/2}\frac{\cos(\Omega_0 t - n\Omega_0 t) - \cos(\Omega_0 t + n\Omega_0 t)}{2}\,dt = \frac{A}{T}\int_0^{T/2}\left[\cos(1-n)\Omega_0 t - \cos(1+n)\Omega_0 t\right]dt$$

$$= \frac{A}{T}\left[\frac{\sin(1-n)\Omega_0 t}{(1-n)\Omega_0} - \frac{\sin(1+n)\Omega_0 t}{(1+n)\Omega_0}\right]_0^{T/2} = \frac{A}{T}\left[\frac{\sin(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}} - \frac{\sin(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}}\right]_0^{T/2}$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$= \frac{A}{T}\left[\frac{\sin(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} - \frac{\sin(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} - \frac{\sin 0}{(1-n)\frac{2\pi}{T}} + \frac{\sin 0}{(1+n)\frac{2\pi}{T}}\right]$$

$\boxed{\sin 0 = 0}$

$$= \frac{A\sin(1-n)\pi}{(1-n)2\pi} - \frac{A\sin(1+n)\pi}{(1+n)2\pi}$$

The above expression for b_n can be evaluated for all values of n except for n = 1. For n = 1, b_n has to be evaluated separately as shown below.

$$b_1 = \frac{2}{T}\int_0^T x(t)\sin\Omega_0 t\,dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\,\sin\Omega_0 t\,dt = \frac{2A}{T}\int_0^{T/2}\sin^2\Omega_0 t\,dt$$

$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$

$$= \frac{2A}{T}\int_0^{T/2}\frac{1-\cos 2\Omega_0 t}{2}\,dt = \frac{A}{T}\int_0^{T/2}(1-\cos 2\Omega_0 t)\,dt = \frac{A}{T}\left[t - \frac{\sin 2\Omega_0 t}{2\Omega_0}\right]_0^{T/2}$$

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$$\therefore b_1 = \frac{A}{T}\left[t - \frac{\sin\frac{4\pi}{T}t}{\frac{4\pi}{T}}\right]_0^{T/2} = \frac{A}{T}\left[\frac{T}{2} - \frac{\sin\frac{4\pi}{T}\frac{T}{2}}{\frac{4\pi}{T}} - 0 + \frac{\sin 0}{\frac{4\pi}{T}}\right]$$

$$= \frac{A}{2} - \frac{A}{4\pi}\sin 2\pi = \frac{A}{2}$$

$\Omega_0 = \frac{2\pi}{T}$; $\sin 0 = 0$; $\sin 2\pi = 0$

$$\therefore b_1 = \frac{A}{2}$$

$$b_n = \frac{A\sin(1-n)\pi}{(1-n)2\pi} - \frac{A\sin(1+n)\pi}{(1+n)2\pi}$$

For integer values of n, except when n = 1, $\sin(1-n)\pi = 0$.

For integer values of n, $\sin(1+n)\pi = 0$.

$\therefore b_n = 0$ for all values of n except n = 1.

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, a_n exists only for even values of n and $b_n = 0$ for all values of n except when n = 1.

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=\text{even}} a_n \cos n\Omega_0 t + b_1 \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + a_2\cos 2\Omega_0 t + a_4\cos 4\Omega_0 t + a_6\cos 6\Omega_0 t + a_8\cos 8\Omega_0 t + \ldots + b_1\sin\Omega_0 t$$

$$= \frac{A}{\pi} - \frac{2A}{3\pi}\cos 2\Omega_0 t - \frac{2A}{15\pi}\cos 4\Omega_0 t - \frac{2A}{35\pi}\cos 6\Omega_0 t - \frac{2A}{63\pi}\cos 8\Omega_0 t - \ldots + \frac{A}{2}\sin\Omega_0 t$$

$$= \frac{A}{\pi} + \frac{2A}{\pi}\left[\frac{\pi}{4}\sin\Omega_0 t - \frac{\cos 2\Omega_0 t}{3} - \frac{\cos 4\Omega_0 t}{15} - \frac{\cos 6\Omega_0 t}{35} - \frac{\cos 8\Omega_0 t}{63} - \ldots\right]$$

Example 4.10

Find the Fourier series of the signal shown in fig 4.10.1.

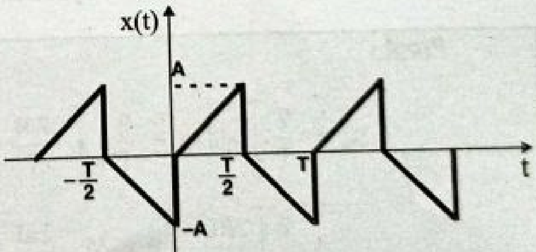

Fig 4.10.1.

Solution

The given signal has half-wave symmetry and so a_0 will be zero.
The Fourier coefficients a_n and b_n will exist only for odd integer values of n.

$$\therefore a_0 = 0 \;;\; a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \;;\; b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt$$

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

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$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

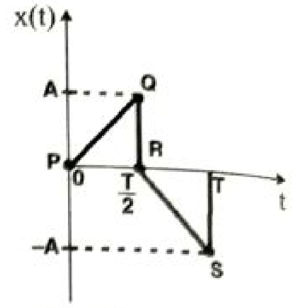

Fig 1.

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\dfrac{T}{2}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-\frac{T}{2}} \Rightarrow x(t) = \frac{2At}{T}$$

Consider points R and S, as shown in fig 1.

Coordinates of point-R = $[t_3, x(t_3)] = \left[\dfrac{T}{2}, 0\right]$

Coordinates of point-S = $[t_4, x(t_4)] = [T, -A]$

On substituting the coordinates of points R and S in equation (1) we get,

$$\frac{x(t) - 0}{0 - (-A)} = \frac{t - \frac{T}{2}}{\frac{T}{2} - T} \Rightarrow \frac{x(t)}{A} = \frac{t - \frac{T}{2}}{-\frac{T}{2}} \Rightarrow \frac{x(t)}{A} = -\frac{2t}{T} + 1 \Rightarrow x(t) = A - \frac{2At}{T}$$

Now the mathematical equation of the waveform is given by,

$$x(t) = \frac{2At}{T} \quad ; \text{ for } t = 0 \text{ to } \frac{T}{2}$$
$$= A - \frac{2At}{T} \quad ; \text{ for } t = \frac{T}{2} \text{ to } T$$

Evaluation of a_0

The given signal has half wave symmetry and so $a_0 = 0$.

Proof :

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\left[\int_0^{T/2} \frac{2At}{T}\,dt + \int_{T/2}^{T}\left(A - \frac{2At}{T}\right)dt\right] = \frac{2}{T}\left[\left[\frac{2At^2}{2T}\right]_0^{T/2} + \left[At - \frac{2At^2}{2T}\right]_{T/2}^{T}\right]$$
$$= \frac{2}{T}\left[\frac{2AT^2}{8T} - 0 + AT - \frac{2AT^2}{2T} - \frac{AT}{2} + \frac{2AT^2}{8T}\right] = \frac{2}{T}\left[\frac{AT}{4} + AT - AT - \frac{AT}{2} + \frac{AT}{4}\right] = 0$$

Evaluation of a_n

The coefficient a_n for a signal with half-wave symmetry is,

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt$$

$$\therefore a_n = \frac{2}{T}\int_0^{T/2} \frac{2At}{T}\cos n\Omega_0 t\,dt + \frac{2}{T}\int_{T/2}^{T}\left(A - \frac{2At}{T}\right)\cos n\Omega_0 t\,dt$$
$$= \frac{4A}{T^2}\int_0^{T/2} t\cos n\Omega_0 t\,dt + \frac{2A}{T}\int_{T/2}^{T}\cos n\Omega_0 t\,dt - \frac{4A}{T^2}\int_{T/2}^{T} t\cos n\Omega_0 t\,dt$$

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$$\therefore a_n = \frac{4A}{T^2}\left[t \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) dt\right]_0^{T/2} + \frac{2A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$- \frac{4A}{T^2}\left[t \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) dt\right]_{T/2}^{T}$$

$\int uv = u\int v - \int\left[du \int v\right]$	
$u = t$	$v = \cos\Omega_0 t$

$$= \frac{4A}{T^2}\left[\frac{t \sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} + \frac{2A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$- \frac{4A}{T^2}\left[\frac{t \sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_{T/2}^{T}$$

$$= \frac{4A}{T^2}\left[\frac{\frac{T}{2}\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} - \frac{0 \times \sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$+ \frac{2A}{T}\left[\frac{\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}}\right]$$

$\boxed{\sin 0 = 0}$
$\boxed{\cos 0 = 1}$

$$- \frac{4A}{T^2}\left[\frac{T\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} - \frac{\frac{T}{2} \times \sin\frac{n2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} - \frac{\cos\frac{n2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= \frac{A}{n\pi}\sin n\pi + \frac{A}{n^2\pi^2}\cos n\pi - \frac{A}{n^2\pi^2} + \frac{A}{n\pi}\sin n2\pi - \frac{A}{n\pi}\sin n\pi$$

$$- \frac{2A}{n\pi}\sin n2\pi - \frac{A}{n^2\pi^2}\cos n2\pi + \frac{A}{n\pi}\sin n\pi + \frac{A}{n^2\pi^2}\cos n\pi$$

For integer n
$\sin n\pi = 0$
$\sin n2\pi = 0$
$\cos n2\pi = 1$

$$= 0 + \frac{A}{n^2\pi^2}\cos n\pi - \frac{A}{n^2\pi^2} + 0 - 0 - 0 - \frac{A}{n^2\pi^2} + 0 + \frac{A}{n^2\pi^2}\cos n\pi$$

$$= \frac{2A}{n^2\pi^2}\cos n\pi - \frac{2A}{n^2\pi^2} = \frac{2A}{n^2\pi^2}(\cos n\pi - 1)$$

When n is even integer, $\cos n\pi = +1$

When n is odd integer, $\cos n\pi = -1$

$\therefore a_n = 0$; for even integer values of n

$= -\frac{4A}{\pi^2 n^2}$; for odd integer values of n

$\therefore a_1 = -\frac{4A}{1^2\pi^2}$; $a_3 = -\frac{4A}{3^2\pi^2}$; $a_5 = -\frac{4A}{5^2\pi^2}$ and so on.

Evaluation of b_n

The coefficient b_n for a signal with half wave symmetry is,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt$$

$$\therefore b_n = \frac{2}{T}\int_0^{T/2}\frac{2At}{T}\sin n\Omega_0 t\, dt + \frac{2}{T}\int_{T/2}^{T}\left(A - \frac{2At}{T}\right)\sin n\Omega_0 t\, dt$$

$$= \frac{4A}{T^2}\int_0^{T/2} t\sin n\Omega_0 t\, dt + \frac{2A}{T}\int_{T/2}^{T}\sin n\Omega_0 t\, dt - \frac{4A}{T^2}\int_{T/2}^{T} t\sin n\Omega_0 t\, dt$$

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$$\therefore b_n = \frac{4A}{T^2}\left[t\times\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1\times\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$-\frac{4A}{T^2}\left[t\times\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1\times\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_{T/2}^{T}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{4A}{T^2}\left[\frac{-t\cos n\Omega_0 t}{n\Omega_0} - \left(\frac{-\sin n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$-\frac{4A}{T^2}\left[\frac{-t\cos n\Omega_0 t}{n\Omega_0} - \left(\frac{-\sin n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_{T/2}^{T}$$

$$= \frac{4A}{T^2}\left[\frac{-\frac{T}{2}\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} + \frac{0\times\cos 0}{n\frac{2\pi}{T}} - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\Omega_0 = \frac{2\pi}{T}$
$\sin 0 = 0$

$$+\frac{2A}{T}\left[-\frac{\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}}\right]$$

$$-\frac{4A}{T^2}\left[\frac{-T\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} + \frac{\frac{T}{2}\times\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= -\frac{A}{n\pi}\cos n\pi + \frac{A}{n^2\pi^2}\sin n\pi - \frac{A}{n\pi}\cos 2n\pi + \frac{A}{n\pi}\cos n\pi + \frac{2A}{n\pi}\cos n2\pi$$

$$+\frac{A}{n^2\pi^2}\sin n2\pi - \frac{A}{n\pi}\cos n\pi + \frac{A}{n^2\pi^2}\sin n\pi$$

$$= -\frac{A}{n\pi}\cos n\pi + 0 - \frac{A}{n\pi} + \frac{A}{n\pi}\cos n\pi + \frac{2A}{n\pi} + 0 - \frac{A}{n\pi}\cos n\pi + 0$$

For integer n
$\sin n\pi = 0$
$\sin n2\pi = 0$
$\cos n2\pi = 1$

$$= \frac{A}{n\pi} - \frac{A}{n\pi}\cos n\pi = \frac{A}{n\pi}(1-\cos n\pi)$$

When n is even integer, $\cos n\pi = +1$

When n is odd integer, $\cos n\pi = -1$

$\therefore b_n = 0$; for even integer values of n.

$= \frac{2A}{n\pi}$; for odd integer values of n.

$\therefore b_1 = \frac{2A}{\pi}$; $b_3 = \frac{2A}{3\pi}$; $b_5 = \frac{2A}{5\pi}$ and so on.

Fourier Series of x(t)

The Fourier series of x(t) is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here $a_0 = 0$ and the Fourier coefficients a_n and b_n exist only for odd values of n.

$$\therefore x(t) = \sum_{n=\text{odd}} a_n \cos n\Omega_0 t + \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5\cos 5\Omega_0 t + \ldots$$

$$+ b_1\sin\Omega_0 t + b_3\sin 3\Omega_0 t + b_5\sin 5\Omega_0 t + \ldots$$

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$$\therefore x(t) = -\frac{4A}{\pi^2}\cos\Omega_0 t - \frac{4A}{3^2\pi^2}\cos 3\Omega_0 t - \frac{4A}{5^2\pi^2}\cos 5\Omega_0 t - .....$$

$$+\frac{2A}{\pi}\sin\Omega_0 t + \frac{2A}{3\pi}\sin 3\Omega_0 t + \frac{2A}{5\pi}\sin 5\Omega_0 t + ...$$

$$= -\frac{4A}{\pi^2}\left(\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right)$$

$$+\frac{2A}{\pi}\left(\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + ...\right)$$

Example 4.11

Determine the exponential form of the Fourier series representation of the signal shown in fig 4.11.1. Hence determine the trigonometric form of Fourier series.

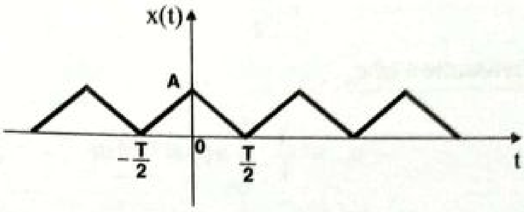

Fig 4.11.1.

Solution

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P, Q and R as shown in fig 1.

Coordinates of point-P $= [t_1, x(t_1)] = \left[\frac{-T}{2}, 0\right]$

Coordinates of point-Q $= [t_2, x(t_2)] = [0, A]$

Coordinates of point-R $= [t_3, x(t_3)] = \left[\frac{T}{2}, 0\right]$

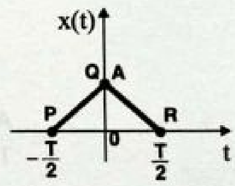

Fig 1.

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t + \frac{T}{2}}{\frac{-T}{2} - 0} \Rightarrow \frac{x(t)}{-A} = \frac{-2t}{T} - 1 \Rightarrow x(t) = A + \frac{2At}{T}$$

On substituting the coordinates of points Q and R in equation (1) we get,

$$\frac{x(t) - A}{A - 0} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{A} - 1 = \frac{-2t}{T} \Rightarrow x(t) = A - \frac{2At}{T}$$

$$\therefore x(t) = A + \frac{2At}{T} \quad ; \quad \text{for } t = -\frac{T}{2} \text{ to } 0$$

$$= A - \frac{2At}{T} \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of c_0

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt$$

When $n = 0$, $c_0 = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^0\, dt = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, dt$

$$= \frac{1}{T}\int_{-T/2}^{0}\left(A + \frac{2At}{T}\right)dt + \frac{1}{T}\int_{0}^{+T/2}\left(A - \frac{2At}{T}\right)dt$$

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$$\therefore\ c_0 = \frac{A}{T}\int_{-T/2}^{0} dt + \frac{2A}{T^2}\int_{-T/2}^{0} t\,dt + \frac{A}{T}\int_{0}^{T/2} dt - \frac{2A}{T^2}\int_{0}^{T/2} t\,dt$$

$$= \frac{A}{T}[t]_{-T/2}^{0} + \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_{-T/2}^{0} + \frac{A}{T}[t]_0^{T/2} - \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_0^{T/2}$$

$$= \frac{A}{T}\left[0+\frac{T}{2}\right] + \frac{2A}{T^2}\left[0-\frac{T^2}{8}\right] + \frac{A}{T}\left[\frac{T}{2}-0\right] - \frac{2A}{T^2}\left[\frac{T^2}{8}-0\right]$$

$$= \frac{A}{2} - \frac{A}{4} + \frac{A}{2} - \frac{A}{4} = \frac{2A}{2} - \frac{2A}{4} = A - \frac{A}{2} = \frac{A}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{0}\left(A + \frac{2At}{T}\right)e^{-jn\Omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2}\left(A - \frac{2At}{T}\right)e^{-jn\Omega_0 t}\,dt$$

$$= \frac{A}{T}\int_{-T/2}^{0} e^{-jn\Omega_0 t}\,dt + \frac{2A}{T^2}\int_{-T/2}^{0} t\,e^{-jn\Omega_0 t}\,dt + \frac{A}{T}\int_{0}^{T/2} e^{-jn\Omega_0 t}\,dt - \frac{2A}{T^2}\int_{0}^{T/2} t\,e^{-jn\Omega_0 t}\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = e^{-jn\Omega_0 t}$

$$= \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1\times\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\,dt\right]_{-T/2}^{0}$$

$$+ \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{0}^{T/2} - \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1\times\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\,dt\right]_{0}^{T/2}$$

$$= \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_{-T/2}^{0} + \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{0}^{+T/2}$$

$$- \frac{2A}{T^2}\left[\frac{t\,e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_{0}^{T/2}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{A}{T}\left[\frac{e^0}{-jn\frac{2\pi}{T}} - \frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-jn\frac{2\pi}{T}}\right] + \frac{2A}{T^2}\left[\frac{0\times e^0}{-jn\frac{2\pi}{T}} - \frac{e^0}{-n^2\frac{4\pi^2}{T^2}} + \frac{T}{2}\frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-n^2\frac{4\pi^2}{T^2}}\right]$$

$$+ \frac{A}{T}\left[\frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-jn\frac{2\pi}{T}} - \frac{e^0}{-jn\frac{2\pi}{T}}\right] - \frac{2A}{T^2}\left[\frac{T}{2}\frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-jn\frac{2\pi}{T}} - \frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-n^2\frac{4\pi^2}{T^2}} - \frac{0\times e^0}{-jn\frac{2\pi}{T}} + \frac{e^0}{-n^2\frac{4\pi^2}{T^2}}\right]$$

$$= -\frac{A}{j2n\pi} + \frac{A\,e^{jn\pi}}{j2n\pi} - 0 + \frac{A}{2n^2\pi^2} - \frac{A\,e^{jn\pi}}{j2n\pi} - \frac{A\,e^{jn\pi}}{2n^2\pi^2} - \frac{A\,e^{-jn\pi}}{j2n\pi}$$

$$+ \frac{A}{j2n\pi} + \frac{A\,e^{-jn\pi}}{j2n\pi} - \frac{A\,e^{-jn\pi}}{2n^2\pi^2} - 0 + \frac{A}{2n^2\pi^2}$$

$$= \frac{A}{n^2\pi^2} - \frac{A\,e^{jn\pi}}{2n^2\pi^2} - \frac{A\,e^{-jn\pi}}{2n^2\pi^2}$$

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We know that,

$$e^{\pm jn\pi} = \cos n\pi \pm j\sin n\pi$$
$$= +1 \pm j0 = 1 \quad ; \text{ for even n}$$
$$= -1 \pm j0 = -1 \quad ; \text{ for odd n.}$$

∴ When n is even,

$$c_n = \frac{A}{n^2\pi^2} - \frac{A}{2n^2\pi^2} - \frac{A}{2n^2\pi^2} = \frac{A}{n^2\pi^2} - \frac{A}{n^2\pi^2} = 0$$

∴ When n is odd,

$$c_n = \frac{A}{n^2\pi^2} + \frac{A}{2n^2\pi^2} + \frac{A}{2n^2\pi^2} = \frac{A}{n^2\pi^2} + \frac{A}{n^2\pi^2} = \frac{2A}{n^2\pi^2}$$

$$\therefore c_{-1} = \frac{2A}{(-1)^2\pi^2} = \frac{2A}{1^2\pi^2} \qquad c_1 = \frac{2A}{1^2\pi^2}$$

$$c_{-3} = \frac{2A}{(-3)^2\pi^2} = \frac{2A}{3^2\pi^2} \qquad c_3 = \frac{2A}{3^2\pi^2}$$

$$c_{-5} = \frac{2A}{(-5)^2\pi^2} = \frac{2A}{5^2\pi^2} \qquad c_5 = \frac{2A}{5^2\pi^2}$$

and so on | and so on

Exponential Form of Fourier Series

The exponential form of Fourier series is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} = \sum_{n=-\infty}^{-1} c_n e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t}$$

Here c_n exist only for odd values of n.

$$\therefore x(t) = \sum_{\substack{n = \text{negative}\\ \text{odd integer}}} c_n e^{jn\Omega_0 t} + c_0 + \sum_{\substack{n = \text{positive}\\ \text{odd integer}}} c_n e^{jn\Omega_0 t}$$

$$= \ldots.. + c_{-5}\, e^{-j5\Omega_0 t} + c_{-3}\, e^{-j3\Omega_0 t} + c_{-1}\, e^{-j\Omega_0 t} + c_0 + c_1\, e^{j\Omega_0 t} + c_3\, e^{j3\Omega_0 t} + c_5\, e^{j5\Omega_0 t} + \ldots..$$

$$x(t) = \ldots.. + \frac{2A}{5^2\pi^2} e^{-j5\Omega_0 t} + \frac{2A}{3^2\pi^2} e^{-j3\Omega_0 t} + \frac{2A}{1^2\pi^2} e^{-j\Omega_0 t} + \frac{A}{2} + \frac{2A}{1^2\pi^2} e^{j\Omega_0 t}$$
$$+ \frac{2A}{3^2\pi^2} e^{j3\Omega_0 t} + \frac{2A}{5^2\pi^2} e^{j5\Omega_0 t} + \ldots..$$

$$= \frac{2A}{\pi^2}\left(\ldots.. + \frac{1}{5^2} e^{-j5\Omega_0 t} + \frac{1}{3^2} e^{-j3\Omega_0 t} + \frac{1}{1^2} e^{-j\Omega_0 t}\right) + \frac{A}{2}$$
$$+ \frac{2A}{\pi^2}\left(\frac{1}{1^2} e^{j\Omega_0 t} + \frac{1}{3^2} e^{j3\Omega_0 t} + \frac{1}{5^2} e^{j5\Omega_0 t} + \ldots..\right)$$

Trigonometric Form of Fourier Series

The trigonometric form of Fourier series can be obtained as shown below.

$$x(t) = \frac{A}{2} + \frac{2A}{\pi^2}\left[\frac{1}{1^2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right) + \frac{1}{3^2}\left(e^{j3\Omega_0 t} + e^{-j3\Omega_0 t}\right) + \frac{1}{5^2}\left(e^{j5\Omega_0 t} + e^{-j5\Omega_0 t}\right) + \ldots..\right]$$

$$= \frac{A}{2} + \frac{2A}{\pi^2}\left[\frac{1}{1^2} 2\cos\Omega_0 t + \frac{1}{3^2} 2\cos 3\Omega_0 t + \frac{1}{5^2} 2\cos 5\Omega_0 t + \ldots..\right]$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{A}{2} + \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \ldots..\right]$$

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Example 4.12

Determine the exponential form of the Fourier series representation of the signal shown in fig 4.12.1. Hence determine the trigonometric form of Fourier series.

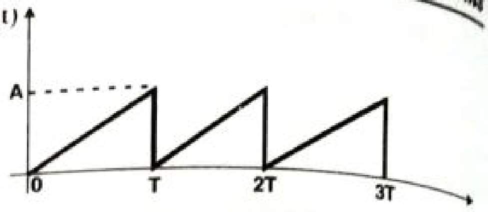

Fig 4.12.1.

Solution

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

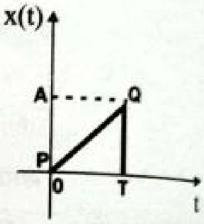

Fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = [T, A]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - T} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-T} \Rightarrow x(t) = \frac{At}{T}$$

$$\therefore x(t) = \frac{At}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

Evaluation of c_0

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt$$

When $n = 0$, $c_0 = \dfrac{1}{T}\displaystyle\int_0^T x(t)\, e^0\, dt = \dfrac{1}{T}\displaystyle\int_0^T x(t)\, dt$

$$= \frac{1}{T}\int_0^T \frac{At}{T}\, dt = \frac{A}{T^2}\int_0^T t\, dt = \frac{A}{T^2}\left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{T^2}{2} - 0\right] = \frac{A}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\int_0^T \frac{At}{T}\, e^{-jn\Omega_0 t}\, dt = \frac{A}{T^2}\int_0^T t\, e^{-jn\Omega_0 t}\, dt$$

$$\boxed{\int uv = u\int v - \int\left[du \int v\right]} \qquad \boxed{u = t \quad v = e^{-jn\Omega_0 t}}$$

$$= \frac{A}{T^2}\left[t\,\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1 \times \frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\, dt\right]_0^T = \frac{A}{T^2}\left[\frac{t\, e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{t\, e^{-jn\frac{2\pi}{T}t}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}t}}{n^2\frac{4\pi^2}{T^2}}\right]_0^T = \frac{A}{T^2}\left[\frac{T\, e^{-jn\frac{2\pi}{T}T}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}T}}{n^2\frac{4\pi^2}{T^2}} - 0 - \frac{e^0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= -\frac{A}{jn2\pi}\, e^{-jn2\pi} + \frac{A}{n^2 4\pi^2}\, e^{-jn2\pi} - \frac{A}{n^2 4\pi^2}$$

$$= -\frac{A}{jn2\pi} + \frac{A}{n^2 4\pi^2} - \frac{A}{n^2 4\pi^2} = -\frac{A}{jn2\pi}$$

$$\boxed{\begin{aligned} e^{-jn2\pi} &= \cos n2\pi - j\sin n2\pi \\ &= 1 - j0 = 1; \text{for integer } n \end{aligned}}$$

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$\therefore\ c_{-1} = \frac{A}{j2\pi}$ | $c_1 = -\frac{A}{j2\pi}$

$c_{-2} = \frac{A}{j4\pi}$ | $c_2 = -\frac{A}{j4\pi}$

$c_{-3} = \frac{A}{j6\pi}$ | $c_3 = -\frac{A}{j6\pi}$

and so on. | and so on.

Exponential Form of Fourier Series

The exponential form of Fourier series is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t} = \sum_{n=-\infty}^{-1} c_n\, e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n\, e^{jn\Omega_0 t}$$

$$= \ldots\ldots c_{-3}\, e^{-j3\Omega_0 t} + c_{-2}\, e^{-j2\Omega_0 t} + c_{-1}\, e^{-j\Omega_0 t} + c_0 + c_1\, e^{j\Omega_0 t} + c_2\, e^{j2\Omega_0 t} + c_3\, e^{j3\Omega_0 t} + \ldots\ldots$$

$$= \ldots\ldots + \frac{A}{j6\pi} e^{-j3\Omega_0 t} + \frac{A}{j4\pi} e^{-j2\Omega_0 t} + \frac{A}{j2\pi} e^{-j\Omega_0 t} + \frac{A}{2} - \frac{A}{j2\pi} e^{j\Omega_0 t} - \frac{A}{j4\pi} e^{j2\Omega_0 t} - \frac{A}{j6\pi} e^{j3\Omega_0 t} \ldots\ldots$$

$$= \frac{A}{j2\pi}\left[\ldots\ldots + \frac{e^{-j3\Omega_0 t}}{3} + \frac{e^{-j2\Omega_0 t}}{2} + \frac{e^{-j\Omega_0 t}}{1}\right] + \frac{A}{2} - \frac{A}{j2\pi}\left[\frac{e^{j\Omega_0 t}}{1} + \frac{e^{j2\Omega_0 t}}{2} + \frac{e^{j3\Omega_0 t}}{3} + \ldots\ldots\right]$$

Trigonometric Form of Fourier Series

The trigonometric form of Fourier series can be obtained as shown below.

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\left[\frac{1}{1}\left(\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\right) + \frac{1}{2}\left(\frac{e^{j2\Omega_0 t} - e^{-j2\Omega_0 t}}{2j}\right) + \frac{1}{3}\left(\frac{e^{j3\Omega_0 t} - e^{-j3\Omega_0 t}}{2j}\right) + \ldots\ldots\right]$$

$$= \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \ldots\ldots\right]$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

4.9 Fourier Transform

4.9.1 Development of Fourier Transform From Fourier Series

The exponential form of Fourier series representation of a periodic signal is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} \qquad \ldots\ldots(4.29)$$

$$\text{where, } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \qquad \ldots\ldots(4.30)$$

In the Fourier representation using equation (4.29), the c_n for various values of n are the spectral components of the signal x(t), located at intervals of fundamental frequency Ω_0. Therefore the frequency spectrum is discrete in nature.

The Fourier representation of a signal using equation (4.29) is applicable for periodic signals. For Fourier representation of non-periodic signals, let us consider that the fundamental period tends to infinity. When the fundamental period tends to infinity, the fundamental frequency Ω_0 tends to zero or becomes very small. Since fundamental frequency Ω_0 is very small, the spectral components will lie very close to each other and so the frequency spectrum becomes continuous.

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In order to obtain the Fourier representation of a non-periodic signal let us consider that the fundamental frequency Ω_0 is very small.

Let, $\Omega_0 \rightarrow \Delta\Omega$

On replacing Ω_0 by $\Delta\Omega$ in equation (4.29) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Delta\Omega t}$$

On substituting for c_n in the above equation from equation (4.30) (by taking τ as dummy variable for integration) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t} \qquad .....(4.31)$$

We know that, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$; $\quad \therefore \frac{1}{T} = \frac{\Omega_0}{2\pi}$

Since $\Omega_0 \rightarrow \Delta\Omega$, $\quad \frac{1}{T} = \frac{\Delta\Omega}{2\pi} \qquad .....(4.32)$

On substituting for $\frac{1}{T}$ from equation (4.32) in equation (4.31) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{\Delta\Omega}{2\pi} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t}\, \Delta\Omega$$

For non-periodic signals, the fundamental period T tends to infinity. On letting limit T tends to infinity in the above equation we get,

$$x(t) = \underset{T\rightarrow\infty}{Lt}\ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t}\, \Delta\Omega$$

When $T \rightarrow \infty$; $\quad \sum \rightarrow \int$; $\quad \Delta\Omega \rightarrow \Omega$

$$\therefore x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau}\, d\tau \right] e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega \qquad .....(4.33)$$

Since τ is a dummy variable, Let $\tau = t$.

$$\text{where, } X(j\Omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau}\, d\tau = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.34)$$

The equation (4.34) is Fourier transform of x(t) and equation (4.33) is inverse Fourier transform of x(t).

Since the equation (4.34) extracts the frequency components of the signal, transformation using equation (4.34) is also called ***analysis*** of the signal x(t). Since the equation (4.33) combines the frequency components of the signal, the inverse transformation using equation (4.33) is also called ***synthesis*** of the signal x(t).

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Definition of Fourier Transform

Let, $x(t)$ = Continuous time signal

$X(j\Omega)$ = Fourier transform of $x(t)$

The Fourier transform of continuous time signal, $x(t)$ is defined as,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

Also, $X(j\Omega)$ is denoted as $\mathcal{F}\{x(t)\}$ where "$\mathcal{F}$" is the symbol used to denote the Fourier transform operation.

$$\therefore\ \mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.35)$$

Note : Sometimes the Fourier transform is expressed as a function of cyclic frequency F, rather than radian frequency Ω. The Fourier transform as a function of cyclic frequency F, is defined as,

$$X(jF) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

Condition for Existence of Fourier Transform

The Fourier transform of $x(t)$ exists if it satisfies the following Dirichlet condition.

1. The $x(t)$ be absolutely integrable.

   i.e., $\int_{-\infty}^{+\infty} x(t)\, dt < \infty$

2. The $x(t)$ should have a finite number of maxima and minima within any finite interval.
3. The $x(t)$ can have a finite number of discontinuities within any interval.

Definition of Inverse Fourier Transform

The ***inverse Fourier transform*** of $X(j\Omega)$ is defined as,

$$x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega \qquad .....(4.36)$$

The signals $x(t)$ and $X(j\Omega)$ are called ***Fourier transform pair*** and can be expressed as shown below,

$$x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} X(j\Omega)$$

Note : When Fourier transform is expressed as a function of cyclic frequency F, the inverse Fourier transform is defined as,

$$x(t) = F^{-1}\{X(jF)\} = \int_{-\infty}^{+\infty} X(jF)\, e^{j2\pi Ft}\, dF$$

4.9.2 Frequency Spectrum Using Fourier Transform

The $X(j\Omega)$ is a complex function of Ω. Hence it can be expressed as a sum of real part and imaginary part as shown below.

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$\therefore X(j\Omega) = X_r(j\Omega) + jX_i(j\Omega)$

where, $X_r(j\Omega)$ = Real part of $X(j\Omega)$

$X_i(j\Omega)$ = Imaginary part of $X(j\Omega)$

The magnitude of $X(j\Omega)$ is called ***Magnitude spectrum.***

$$\therefore \text{Magnitude spectrum, } |X(j\Omega)| = \sqrt{X_r^2(j\Omega) + X_i^2(j\Omega)} \quad .....(4.37)$$

(or)

$$\text{Magnitude spectrum, } |X(j\Omega)| = \sqrt{X(j\Omega)\ X^*(j\Omega)} \quad .....(4.38)$$

where, $X^*(j\Omega)$ = Conjugate of $X(j\Omega)$

The phase of $X(j\Omega)$ is called ***Phase spectrum.***

$$\therefore \text{Phase spectrum, } \angle X(j\Omega) = \tan^{-1}\frac{X_i(j\Omega)}{X_r(j\Omega)} \quad .....(4.39)$$

The magnitude spectrum will always have even symmetry and phase spectrum will have odd symmetry. The magnitude and phase spectrum together called ***frequency spectrum.***

4.10 Properties of Fourier Transform

1. Linearity

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

The linearity property of Fourier transform says that,

$$\mathcal{F}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, X_1(j\Omega) + a_2\, X_2(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X_1(j\Omega) = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \quad \text{and} \quad X_2(j\Omega) = \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt \quad .....(4.40)$$

Consider the linear combination $a_1\, x_1(t) + a_2\, x_2(t)$. On taking Fourier transform of this signal we get,

$$\mathcal{F}\{a_1\, x_1(t) + a_2\, x_2(t)\} = \int_{-\infty}^{+\infty} [a_1\, x_1(t) + a_2\, x_2(t)]\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} a_1\, x_1(t)\, e^{-j\Omega t}\, dt + \int_{-\infty}^{+\infty} a_2\, x_2(t)\, e^{-j\Omega t}\, dt$$

$$= a_1 \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt + a_2 \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt$$

Using equation (4.40)

$$= a_1\, X_1(j\Omega) + a_2\, X_2(j\Omega)$$

2. Time shifting

The time shifting property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(t - t_0)\} = e^{-j\Omega t_0} X(j\Omega)$$

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Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.41)$$

$$\therefore \mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega(\tau + t_0)}\, d\tau$$

Let, $t - t_0 = \tau$
$\therefore t = \tau + t_0$
On differentiating
$dt = d\tau$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau} \times e^{-j\Omega t_0}\, d\tau = e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau}\, d\tau$$

Since τ is a dummy variable for ingetration we can change τ to t.

$$= e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = e^{-j\Omega t_0} X(j\Omega)$$

Using equation (4.41)

3. Time scaling

The time scaling property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore \mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\left(\frac{\tau}{a}\right)} \frac{d\tau}{a}$$

Put, $at = \tau$; $\therefore t = \frac{\tau}{a}$; $dt = \frac{d\tau}{a}$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau = \frac{1}{a} X\left(\frac{j\Omega}{a}\right)$$

The term $\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau$ is similar to the form of Fourier transform except that Ω is replaced by $\left(\frac{\Omega}{a}\right)$.

$\therefore \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau = X\left(\frac{j\Omega}{a}\right)$

The above transform is applicable for positive values of "a".

If "a" happens to be negative then it can be proved that,

$$\mathcal{F}\{x(at)\} = -\frac{1}{a} X\left(\frac{j\Omega}{a}\right)$$

Hence in general,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right) \text{ for both positive and negative values of "a"}$$

4. Time reversal

The time reversal property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(-t)\} = X(-j\Omega)$$

Proof :

From time scaling property we know that,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$$

Let, $a = -1$.

$$\therefore \mathcal{F}\{x(-t)\} = X(-j\Omega)$$

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5. Conjugation

The conjugation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x^*(t)\} = X^*(-j\Omega)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore \mathcal{F}\{x^*(t)\} = \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\Omega t}\, dt$$

$$= \left[\int_{-\infty}^{+\infty} x(t)\, e^{j\Omega t}\, dt\right]^* = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t}\, dt\right]^*$$

$$= [X(-j\Omega)]^* = X^*(-j\Omega)$$

The term, $\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $-\Omega$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t} dt = X(-j\Omega)$$

6. Frequency shifting

The frequency shifting property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{e^{j\Omega_0 t}\, x(t)\} = X(j(\Omega - \Omega_0))$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore \mathcal{F}\{e^{j\Omega_0 t}\, x(t)\} = \int_{-\infty}^{+\infty} \left[e^{j\Omega_0 t}\, x(t)\right] e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(t) e^{j\Omega_0 t}\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j(\Omega - \Omega_0)t}\, dt = X(j(\Omega - \Omega_0))$$

The term $\int_{-\infty}^{+\infty} x(t)\, e^{-j(\Omega-\Omega_0)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $\Omega - \Omega_0$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\, e^{-j(\Omega-\Omega_0)t} dt = X(j(\Omega - \Omega_0))$$

7. Time differentiation

The differentiation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = j\Omega\, X(j\Omega)$$

Proof :

Consider the definition of Fourier transform of x(t).

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \quad \text{.....(4.42)}$$

$$\therefore \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \int_{-\infty}^{+\infty} \left(\frac{d}{dt}x(t)\right) e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} e^{-j\Omega t}\left(\frac{d}{dt}x(t)\right) dt$$

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$$\therefore \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \left[e^{-j\Omega t}\ x(t)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}(-j\Omega)\ e^{-j\Omega t}\ x(t)\ dt$$

$\int uv = u\int v - \int\left[du\int v\right]$

$$= e^{-\infty}\ x(\infty) - e^{+\infty}\ x(-\infty) + j\Omega\int_{-\infty}^{+\infty} x(t)\ e^{-j\Omega t}\ dt$$

$x(-\infty) = 0$
$e^{-\infty} = 0$

$$= j\Omega\int_{-\infty}^{+\infty} x(t)\ e^{-j\Omega t}\ dt = j\Omega\ X(j\Omega)$$

Using equation (4.42)

8. Time integration

The integration property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ and $X(0) = 0$ then

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\ d\tau\right\} = \frac{1}{j\Omega}X(j\Omega)$$

Proof :

Consider a continuous time signal x(t). Let X(jΩ) be Fourier transform of x(t). Since integration and differentiation are inverse operations, x(t) can be expressed as shown below.

$$\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\ d\tau\right] = x(t)$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\ d\tau\right]\right\} = \mathcal{F}\{x(t)\}$$

$$j\Omega\ \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\ d\tau\right\} = \mathcal{F}\{x(t)\}$$

Using time differentiation property of Fourier transform.

$\mathcal{F}\{x(t)\} = X(j\Omega)$

$$\therefore \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\ d\tau\right\} = \frac{1}{j\Omega}X(j\Omega)$$

9. Frequency differentiation

The frequency differentiation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$, then

$$\mathcal{F}\{t\,x(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\ e^{-j\Omega t}\ dt$$

On differentiating the above equation with respect to Ω we get,

$$\frac{d}{d\Omega}X(j\Omega) = \frac{d}{d\Omega}\left(\int_{-\infty}^{+\infty} x(t)\ e^{-j\Omega t}\ dt\right)$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{d\Omega}e^{-j\Omega t}\right) dt$$

Interchanging the order of integration and differentiation

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$$\therefore \frac{d}{d\Omega}X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\left(-jt\,e^{-j\Omega t}\right)dt = \frac{1}{j}\int_{-\infty}^{+\infty}(t\,x(t))\,e^{-j\Omega t}\,dt$$

$$= \frac{1}{j}\mathcal{F}\{t\,x(t)\}$$

$$\therefore \mathcal{F}\{t\,x(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

$-j = -j\times\frac{j}{j} = \frac{1}{j}$

Using definition of Fourier transform.

10. *Convolution theorem*

The convolution theorem of Fourier transform says that, Fourier transform of convolution of two signals is given by the product of the Fourier transform of the individual signals.

i.e., if $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$ then,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\Omega)\,X_2(j\Omega) \quad .....(4.43)$$

The equation (4.43) is also known as convolution property of Fourier transform.

With reference to chapter-2, section -2.9 we get,

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau \quad .....(4.44)$$

where τ is a dummy variable used for integration.

Proof :

Let $x_1(t)$ and $x_2(t)$ be two time domain signals. Now, by definition of Fourier transform,

$$X_1(j\Omega) = \mathcal{F}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\,e^{-j\Omega t}\,dt \quad .....(4.45)$$

$$X_2(j\Omega) = \mathcal{F}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\,e^{-j\Omega t}\,dt \quad .....(4.46)$$

Using definition of Fourier transform we can write,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\left[x_1(t) * x_2(t)\right]e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{+\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau\right]e^{-j\Omega t}\,dt \quad .....(4.47)$$

Let, $e^{-j\Omega t} = e^{j\Omega\tau}\times e^{-j\Omega\tau}\times e^{-j\Omega t} = e^{-j\Omega\tau}\times e^{-j\Omega(t-\tau)} = e^{-j\Omega\tau}\times e^{-j\Omega M}$ .....(4.48)

where, $M = t-\tau$ and so, $dM = dt$ .....(4.49)

Using equations (4.48) and (4.49), the equation (4.47) can be written as,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1(\tau)\,x_2(M)\,e^{-j\Omega\tau}\,e^{-j\Omega M}\,d\tau\,dM$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\,e^{-j\Omega\tau}\,d\tau \times \int_{-\infty}^{+\infty} x_2(M)\,e^{-j\Omega M}\,dM \quad .....(4.50)$$

In equation (4.50), τ and M are dummy variables used for integration, and so they can be changed to t. Therefore equation (4.50) can be written as,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty} x_1(t)\,e^{-j\Omega t}\,dt \times \int_{-\infty}^{+\infty} x_2(t)\,e^{-j\Omega t}\,dt$$

$$= X_1(j\Omega)\,X_2(j\Omega)$$

Using equations (4.45) and (4.46)

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11. Frequency convolution

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$.

The frequency convolution property of Fourier transform says that,

$$\mathcal{F}\{x_1(t)\, x_2(t)\} = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\lambda$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore\ \mathcal{F}\{x_1(t)x_2(t)\} = \int_{t=-\infty}^{t=+\infty} x_1(t)x_2(t)\, e^{-j\Omega t}\, dt \qquad .....(4.51)$$

By the definition of inverse Fourier transform we get,

$$x_1(t) = \mathcal{F}^{-1}\{X_1(j\Omega)\} = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X_1(j\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, e^{j\lambda t}\, d\lambda \qquad .....(4.52)$$

Here Ω is the vairable used for ingetration. Let us change Ω to λ.

On substituting for $X_1(t)$ from equation (4.52) in equation (4.51) we get,

$$\mathcal{F}\{x_1(t)x_2(t)\} = \int_{t=-\infty}^{t=+\infty} \left[\frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, e^{j\lambda t}\, d\lambda\right] x_2(t)\, e^{-j\Omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j\Omega t}\, e^{j\lambda t}\, dt\right] d\lambda$$

Interchanging the order of integration.

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j(\Omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\lambda$$

The term, $\int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j(\Omega-\lambda)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $\Omega-\lambda$.

$$\therefore \int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j(\Omega-\lambda)t}\, dt = X_2(j(\Omega-\lambda))$$

12. Parseval's relation

The Parseval's relation says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\Omega)|^2\, d\Omega$$

Proof :

Let $x(t)$ be a continuous time signal and $x^*(t)$ be conjugate of $x(t)$.

Now, $|x(t)|^2 = x(t)\, x^*(t)$

On integrating the above equation with respect to t we get,

$$\int_{t=-\infty}^{t=+\infty} |x(t)|^2\, dt = \int_{t=-\infty}^{t=+\infty} x(t)\, x^*(t)\, dt \qquad .....(4.53)$$

By definition of inverse Fourier transform, we can write,

$$x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

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On taking conjugate of the above equation we get,

$$x^*(t) = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, e^{-j\Omega t}\, d\Omega \qquad .....(4.54)$$

Using equation (4.54) the equation (4.53) can be written as,

$$\int_{t=-\infty}^{t=+\infty} |x(t)|^2\, dt = \int_{t=-\infty}^{t=+\infty} x(t)\left[\frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, e^{-j\Omega t}\, d\Omega\right] dt$$

Interchanging the order of integration.

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\left[\int_{t=-\infty}^{t=+\infty} x(t)\, e^{-j\Omega t}\, dt\right] d\Omega$$

Using definition of Fourier transform.

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, X(j\Omega)\, d\Omega$$

$X(j\Omega)\, X^*(j\Omega) = |X(j\Omega)|^2$

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} |X(j\Omega)|^2\, d\Omega$$

Note : *The term $|X(j\Omega)|^2$ represents the distribution of energy as function of Ω and so it is called* ***energy density spectrum*** *or* ***energy spectral density*** *of the signal $x(t)$.*

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Table 4.3 : Summary of Properties of Fourier Transform

Let, $\mathcal{F}\{x(t)\} = X(j\Omega)$; $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

Property	Time domain signal	Frequency domain signal
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$
Time shifting	$x(t - t_0)$	$e^{-j\Omega t_0} X(j\Omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\Omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\Omega)$
Conjugation	$x^*(t)$	$X^*(-j\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(j(\Omega - \Omega_0))$
Time differentiation	$\frac{d}{dt} x(t)$	$j\Omega X(j\Omega)$
Time integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(j\Omega)}{j\Omega} = \pi X(0)\,\delta(\Omega)$
Frequency differentiation	$t\, x(t)$	$j \frac{d}{d\Omega} X(j\Omega)$
Time convolution	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau) d\tau$	$X_1(j\Omega)\, X_2(j\Omega)$
Frequency convolution (or Multiplication)	$x_1(t)\, x_2(t)$	$\frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega - \lambda))\, d\lambda$
Symmetry of real signals	x(t) is real	$X(j\Omega) = X^*(j\Omega)$ $\lvert X(j\Omega)\rvert = \lvert X(-j\Omega)\rvert$; $\angle X(j\Omega) = -\angle X(-j\Omega)$ $Re\{X(j\Omega)\} = Re\{X(-j\Omega)\}$ $Im\{X(j\Omega)\} = -Im\{X(-j\Omega)\}$
Real and even	x(t) is real and even	$X(j\Omega)$ are real and even
Real and odd	x(t) is real and odd	$X(j\Omega)$ are imaginary and odd
Duality	If $x_2(t) \equiv X_1(j\Omega)$ [i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions] then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$ [i.e., $X_2(j\Omega)$ and $2\pi x_1(-j\Omega)$ are similar functions]	
Area under a frequency domain signal	$\int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$	
Area under a time domain signal	$\int_{-\infty}^{+\infty} x(t)\, dt = X(0)$	
Parseval's relation	Energy in time domain is, $E = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$	Energy in frequency domain is, $E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2 d\Omega$
	$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\, d\Omega$	

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4.11 Fourier Transform of Some Important Signals

Fourier Transform of Unit Impulse Signal

The impulse signal is defined as,

$$x(t) = \delta(t) = \infty \; ; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0 \; ; \; t \neq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\Omega t}\, dt$$

$$= 1 \times e^{-j\Omega t}\Big|_{t=0} = 1 \times e^{0} = 1$$

δ(t) exists only for t = 0

$$\therefore \boxed{\mathcal{F}\{x(t)\} = 1}$$

The plot of impulse signal and its magnitude spectrum are shown in fig 4.18 and fig 4.19 respectively.

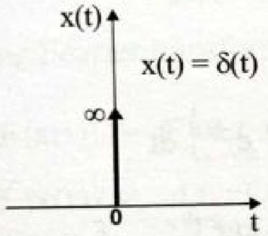

Fig 4.18 : Impulse signal.

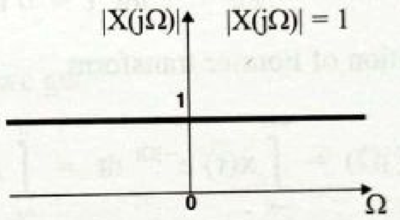

Fig 4.19 : Magnitude spectrum of impulse signal.

Fourier Transform of Single Sided Exponential Signal

The single sided exponential signal is defined as,

$$x(t) = A\, e^{-at} \quad ; \quad \text{for } t \geq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{0}^{+\infty} A\, e^{-at}\, e^{-j\Omega t}\, dt$$

$$= \int_{0}^{+\infty} A\, e^{-(a+j\Omega)t}\, dt = \left[\frac{A\, e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{+\infty}$$

$$= \left[\frac{A\, e^{-\infty}}{-(a+j\Omega)} - \frac{A\, e^{0}}{-(a+j\Omega)}\right] = \frac{A}{a+j\Omega}$$

$e^{-\infty} = 0$

$$\therefore \boxed{\mathcal{F}\{A\, e^{-at}\, u(t)\} = \frac{A}{a+j\Omega}} \quad .....(4.56)$$

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The plot of exponential signal and its magnitude spectrum are shown in fig 4.20 and fig 4.21 respectively.

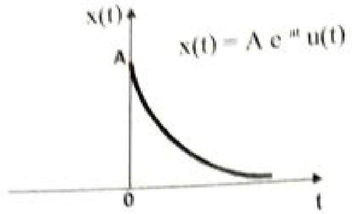

Fig 4.20: Single sided exponential signal.

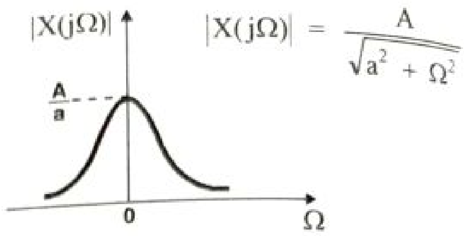

Fig 4.21 : Magnitude spectrum of single sided exponential signal.

Fourier Transform of Double Sided Exponential Signal

The double sided exponential signal is defined as,

$$x(t) = A\,e^{-a|t|} \quad ; \text{ for all } t$$

$$\therefore\ x(t) = A\,e^{+at} \quad ; \text{ for } t = -\infty \text{ to } 0$$

$$= A\,e^{-at} \quad ; \text{ for } t = 0 \text{ to } +\infty$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{-\infty}^{0} A\,e^{at}\,e^{-j\Omega t}\,dt + \int_{0}^{+\infty} A\,e^{-at}\,e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{0} A\,e^{(a-j\Omega)t}\,dt + \int_{0}^{\infty} A\,e^{-(a+j\Omega)t}\,dt = \left[\frac{A\,e^{(a-j\Omega)t}}{a-j\Omega}\right]_{-\infty}^{0} + \left[\frac{A\,e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_{0}^{\infty}$$

$$= \frac{A\,e^{0}}{a-j\Omega} - \frac{A\,e^{-\infty}}{a-j\Omega} + \frac{A\,e^{-\infty}}{-(a+j\Omega)} - \frac{A\,e^{0}}{-(a+j\Omega)} = \frac{A}{a-j\Omega} + \frac{A}{a+j\Omega}$$

$$= \frac{A(a+j\Omega) + A(a-j\Omega)}{(a-j\Omega)\,(a+j\Omega)} = \frac{2aA}{a^2+\Omega^2}$$

$e^{-\infty} = 0$

$(a+b)(a-b) = a^2 - b^2$ | $j^2 = -1$

$$\therefore\ \mathcal{F}\left\{A\,e^{-a|t|}\right\} = \frac{2aA}{a^2+\Omega^2} \qquad .....(4.57)$$

The plot of double sided exponential signal and its magnitude spectrum are shown in fig 4.22 and fig 4.23 respectively.

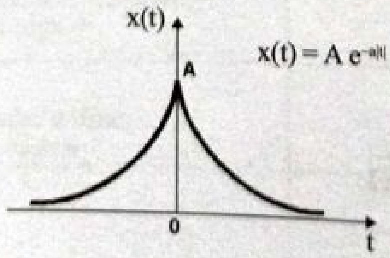

Fig 4.22 : Double sided exponential signal.

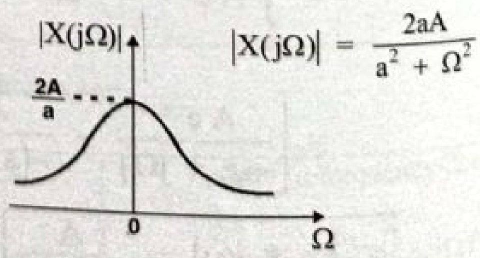

Fig 4.23 : Magnitude spectrum of double sided exponential signal.

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Fourier Transform of a Constant

Let, $x(t) = A$, where A is a constant.

If definition of Fourier transform is directly applied, the constant will not satisfy the condition,

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

Hence the constant can be viewed as a double sided exponential with limit "a" tends to 0 as shown below.

Let $x_1(t)$ = Double sided exponential signal.

The double sided exponential signal is defined as,

$$x_1(t) = A\, e^{-a|t|}$$

$$\text{i.e.,} \quad x_1(t) = A\, e^{at} \;;\; \text{for } t = -\infty \text{ to } 0$$
$$= A\, e^{-at} \;;\; \text{for } t = 0 \text{ to } +\infty$$

$$\therefore\; x(t) = A \underset{a \to 0}{\text{Lt}}\; x_1(t)$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\underset{a \to 0}{\text{Lt}}\; x_1(t)\right\}$$

$$\mathcal{F}\{x(t)\} = \underset{a \to 0}{\text{Lt}}\; \mathcal{F}\{x_1(t)\}$$

$$\boxed{\mathcal{F}\{x(t)\} = X(j\Omega) \;\Big|\; \mathcal{F}\{x_1(t)\} = X_1(j\Omega)}$$

$$X(j\Omega) = \underset{a \to 0}{\text{Lt}}\; [X_1(j\Omega)]$$

$$= \underset{a \to 0}{\text{Lt}}\; \frac{2aA}{\Omega^2 + a^2}$$

Using equation (4.57)

The above equation is 0 for all values of Ω except at $\Omega = 0$.

At $\Omega = 0$, the above equation represents an impulse of magnitude "k".

$$\therefore\; X(j\Omega) = k\,\delta(\Omega) \;;\; \Omega = 0$$
$$= 0 \;;\; \Omega \neq 0$$

The magnitude "k" can be evaluated as shown below.

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

$$k = \int_{-\infty}^{+\infty} \frac{2aA}{\Omega^2 + a^2}\, d\Omega = 2aA \int_{-\infty}^{+\infty} \frac{1}{\Omega^2 + a^2}\, d\Omega$$

$$= 2aA\left[\frac{1}{a}\tan^{-1}\left(\frac{\Omega}{a}\right)\right]_{-\infty}^{+\infty} = 2aA\left[\frac{1}{a}\tan^{-1}(+\infty) - \frac{1}{a}\tan^{-1}(-\infty)\right]$$

$$= 2aA\left[\frac{1}{a}\frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right)\right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A \qquad .....(4.58)$$

$$\therefore\; \boxed{\mathcal{F}\{A\} = 2\pi A\,\delta(\Omega)}$$

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The plot of constant and its magnitude spectrum are shown in fig 4.24 and fig 4.25 respectively.

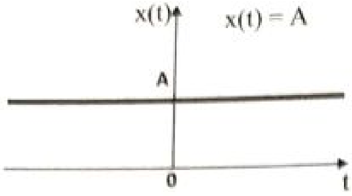

Fig 4.24 : Constant.

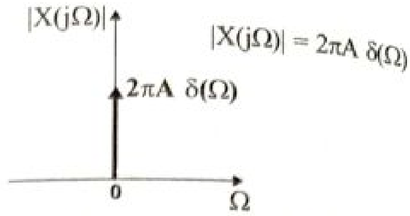

Fig 4.25 : Magnitude spectrum of constant.

Fourier Transform of Signum Function

The signum function is defined as,

$$x(t) = \mathrm{sgn}(t) = \begin{cases} 1 & ; \ t > 0 \\ -1 & ; \ t < 0 \end{cases}$$

The signum function can be expressed as a sum of two one sided exponential signal and taking limit "a" tends to 0 as shown below.

$$\therefore \ \mathrm{sgn}(t) = \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at}\,u(t) - e^{at}\,u(-t)\right]$$

$$\therefore \quad x(t) = \mathrm{sgn}(t) = \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at}\,u(t) - e^{at}\,u(-t)\right]$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{-\infty}^{+\infty} \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at}\,u(t) - e^{at}\,u(-t)\right] e^{-j\Omega t}\,dt$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\int_{0}^{+\infty} e^{-at}\,e^{-j\Omega t}\,dt - \int_{-\infty}^{0} e^{at}\,e^{-j\Omega t}\,dt\right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\int_{0}^{+\infty} e^{-(a + j\Omega)t}\,dt - \int_{-\infty}^{0} e^{+(a - j\Omega)t}\,dt\right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\left[\frac{e^{-(a + j\Omega)t}}{-(a + j\Omega)}\right]_{0}^{\infty} - \left[\frac{e^{(a - j\Omega)t}}{(a - j\Omega)}\right]_{-\infty}^{0}\right] \qquad \boxed{e^{0} = 1\ ;\ e^{-\infty} = 0}$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\frac{e^{-\infty}}{-(a + j\Omega)} - \frac{e^{0}}{-(a + j\Omega)} - \frac{e^{0}}{a - j\Omega} + \frac{e^{-\infty}}{a - j\Omega}\right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\frac{1}{a + j\Omega} - \frac{1}{a - j\Omega}\right] = \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

$$\therefore \ \boxed{\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\Omega}} \qquad .....(4.59)$$

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The plot of signum function and its magnitude spectrum are shown in fig 4.26 and fig 4.27 respectively.

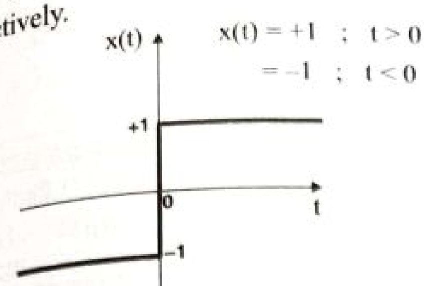

Fig 4.26 : Signum function.

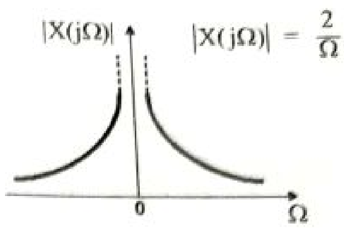

Fig 4.27 : Magnitude spectrum of signum function.

Fourier Transform of Unit Step Signal

The unit step signal is defined as,

$$u(t) = 1 \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

If can be proved that, $\text{sgn}(t) = 2u(t) - 1 \quad \Rightarrow \quad u(t) = \frac{1}{2}[1 + \text{sgn}(t)]$

$$\therefore \; x(t) = u(t) = \frac{1}{2}[1 + \text{sgn}(t)]$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{1}{2}[1 + \text{sgn}(t)]\right\}$$

$$\therefore \; X(j\Omega) = \mathcal{F}\left\{\frac{1}{2}\right\} + \mathcal{F}\left\{\frac{1}{2}\,\text{sgn}(t)\right\} = \frac{1}{2}\,\mathcal{F}\{1\} + \frac{1}{2}\,\mathcal{F}\{\text{sgn}(t)\}$$

$$= \frac{1}{2}\,[2\pi\,\delta(\Omega)] + \frac{1}{2}\left[\frac{2}{j\Omega}\right] = \pi\,\delta(\Omega) + \frac{1}{j\Omega}$$

Using equations (4.58) and (4.59)

$$\therefore \; \boxed{\mathcal{F}\{u(t)\} = \pi\,\delta(\Omega) + \frac{1}{j\Omega}} \quad .....(4.60)$$

The plot of unit step signal and its magnitude spectrum are shown in fig 4.28 and fig 4.29 respectively.

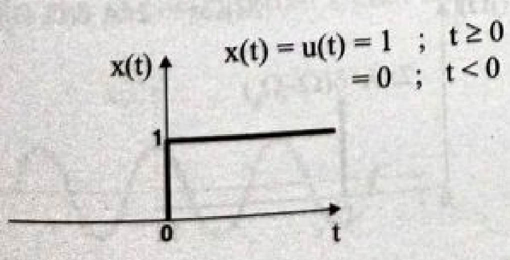

Fig 4.28 : Unit step signal.

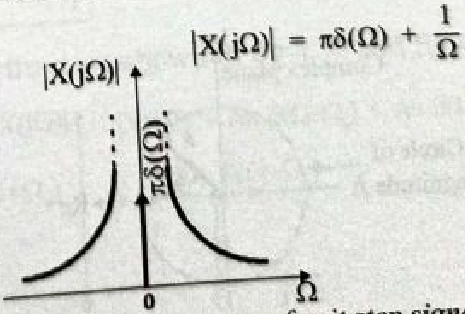

Fig 4.29 : Magnitude spectrum of unit step signal.

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Fourier Transform of Sinusoidal Signal

The sinusoidal signal is defined as,

$$x(t) = A \sin\Omega_0 t = \frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2j}\left[\mathcal{F}\{e^{j\Omega_0 t}\} - \mathcal{F}\{e^{-j\Omega_0 t}\}\right]$$

Using equations (4.61) and (4.62).

$$= \frac{A}{2j}\left[2\pi\,\delta(\Omega - \Omega_0) - 2\pi\,\delta(\Omega + \Omega_0)\right] = \frac{A\pi}{j}\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]$$

$$\therefore \boxed{\mathcal{F}\{A \sin\Omega_0 t\} = \frac{A\pi}{j}\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]} \quad .....(4.63)$$

The plot of sinusoidal signal and its spectrum are shown in fig 4.34 and fig 4.35.

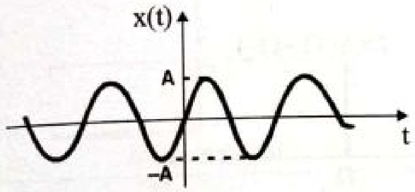

Fig 4.34 : Sinusoidal signal.

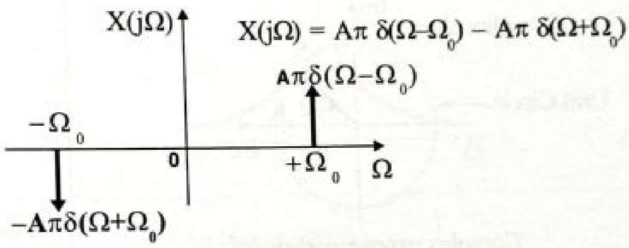

Fig 4.35 : Spectrum of sinusoidal signal.

Fourier Transform of Cosinusoidal Signal

The cosinusoidal signal is defined as,

$$x(t) = A \cos\Omega_0 t = \frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2}\left[\mathcal{F}\{e^{j\Omega_0 t}\} + \mathcal{F}\{e^{-j\Omega_0 t}\}\right]$$

Using equations (4.61) and (4.62).

$$= \frac{A}{2}\left[2\pi\,\delta(\Omega - \Omega_0) + 2\pi\,\delta(\Omega + \Omega_0)\right] = A\pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]$$

$$\therefore \boxed{\mathcal{F}\{A \cos\Omega_0 t\} = A\pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]} \quad .....(4.64)$$

The plot of cosinusoidal signal and its magnitude spectrum are shown in fig 4.36 and fig 4.37.

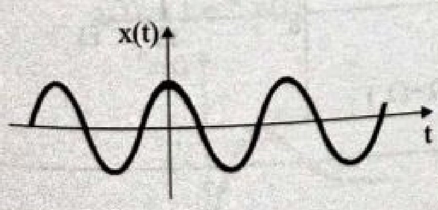

Fig 4.36 : Cosinusoidal signal.

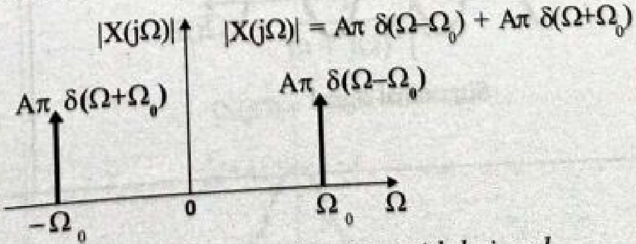

Fig 4.37 : Magnitude spectrum of cosinusoidal signal.

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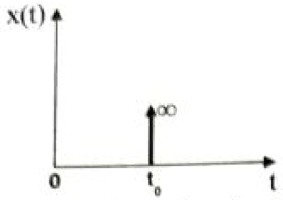
Table 4.4 : Fourier Transform of Standard Signals and their Magnitude Spectrum

<table>
<tr><th>x(t)</th><th>X(jΩ) and Magnitude Spectrum</th></tr>
<tr><td>

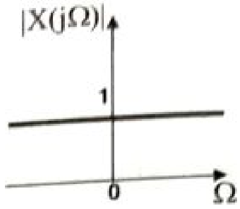
$x(t) = \delta(t-t_0)$



Shifted impulse signal

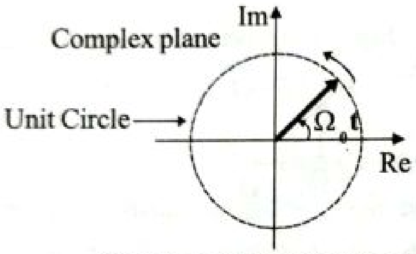
</td><td>

$X(j\Omega) = e^{-j\Omega t_0}$



</td></tr>
<tr><td>

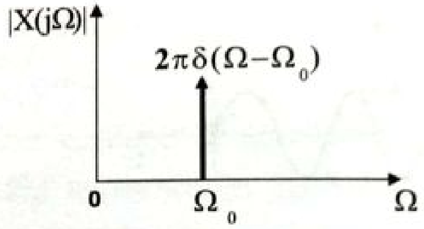
$x(t) = e^{j\Omega_0 t}$



Complex exponential signal

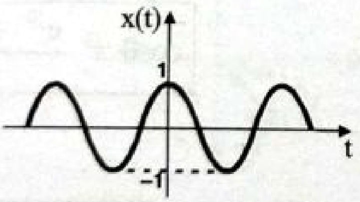
</td><td>

$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$



</td></tr>
<tr><td>

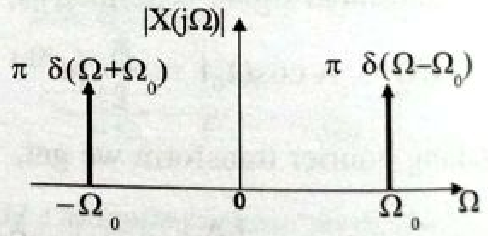
$x(t) = \cos\Omega_0 t$



Cosinusoidal signal

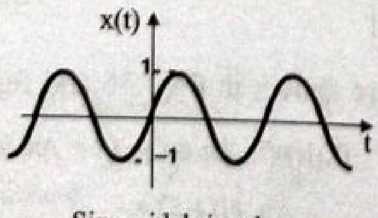
</td><td>

$X(j\Omega) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$



</td></tr>
<tr><td>

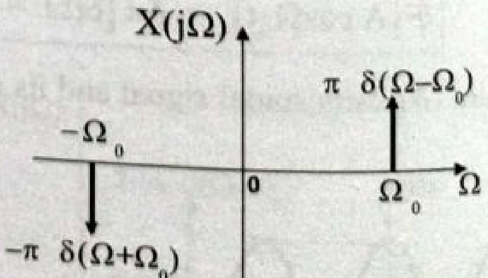
$x(t) = \sin\Omega_0 t$



Sinusoidal signal

</td><td>

$X(j\Omega) = \frac{\pi}{j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$



</td></tr>
</table>

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Table 4.4 : Continued.....

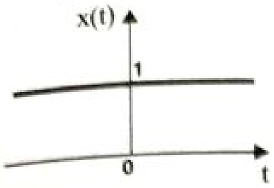
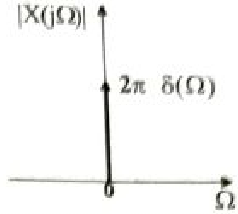
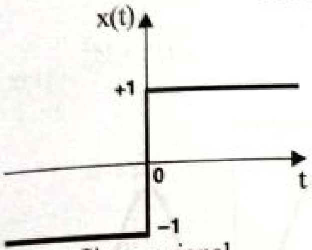
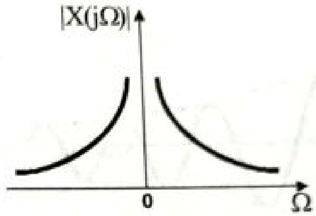
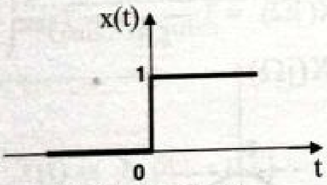
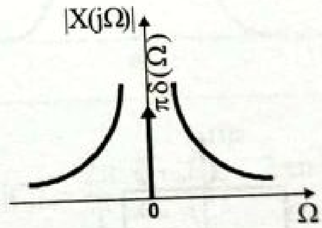
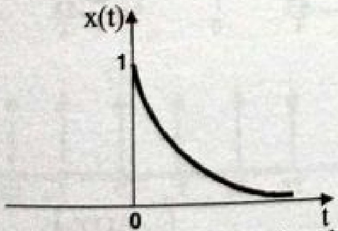
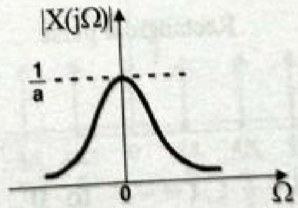
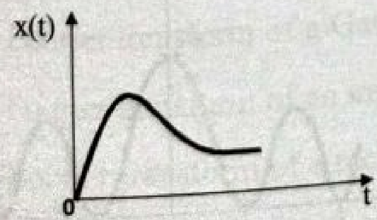
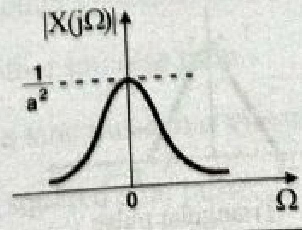
x(t)	X(jΩ) and Magnitude Spectrum
$x(t) = 1$  Constant	$X(j\Omega) = 2\pi\delta(\Omega)$ 
$x(t) = sgn(t) = \frac{t}{\lvert t \rvert} = 1 \;;t > 0$ $= -1 \;;t < 0$  Signum signal	$X(j\Omega) = \frac{2}{j\Omega}$ 
$x(t) = u(t) = 1 \;;\; t \geq 0$ $= 0 \;;\; t < 0$  Unit step signal	$X(j\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega}$ 
$x(t) = e^{-at}\, u(t)$  Decaying exponential signal	$X(j\Omega) = \frac{1}{a + j\Omega}$ 
$x(t) = t\, e^{-at}\, u(t)$  Product of ramp and decaying exponential signal	$X(j\Omega) = \frac{1}{(a + j\Omega)^2}$ 

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Table 4.4 : Continued.....

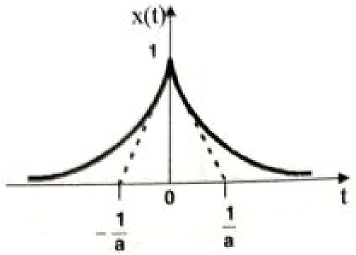
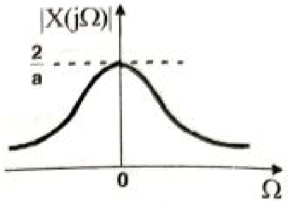
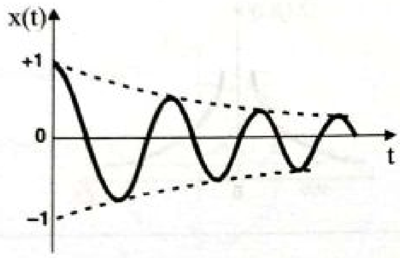
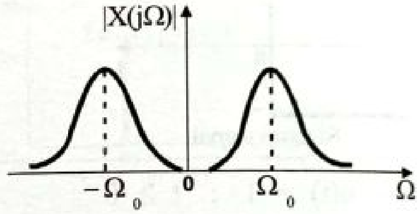
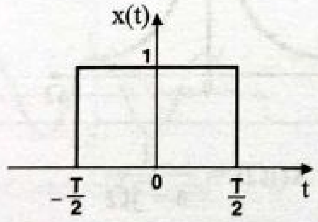
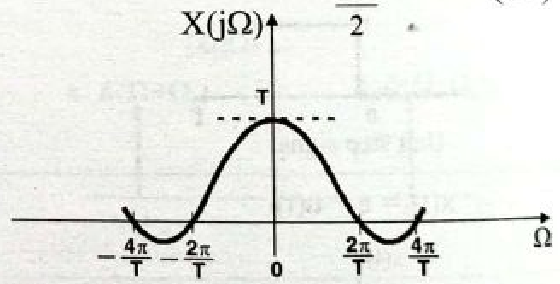
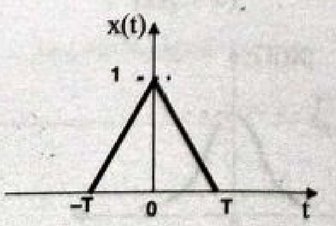
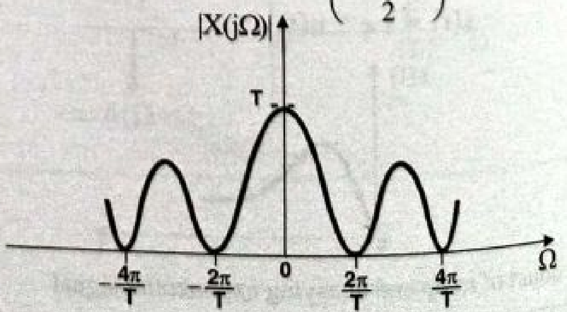
x(t)	X(jΩ) and Magnitude Spectrum
$x(t) = e^{-a\|t\|}$  Double exponential signal	$X(j\Omega) = \frac{2a}{a^2 + \Omega^2}$ 
 Exponentially decaying cosinusoidal signal	$X(j\Omega) = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$ 
 Rectangular pulse	$X(j\Omega) = T\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}} = T\,\text{sinc}\left(\frac{\Omega T}{2}\right)$ 
$x(t) = 1 + \frac{t}{T}\ ;\ t = -T \text{ to } 0$ $= 1 - \frac{t}{T}\ ;\ t = 0 \text{ to } 0$  Triangular pulse	$X(j\Omega) = T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2$ 

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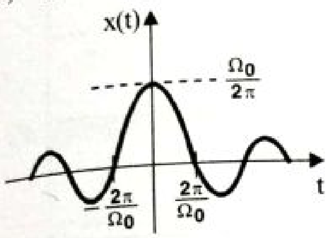
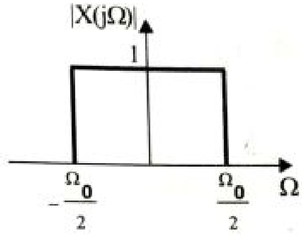
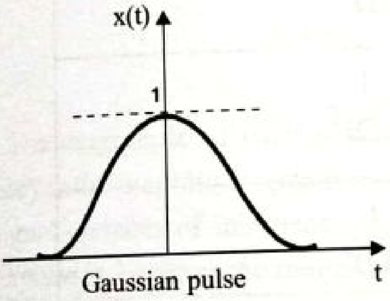
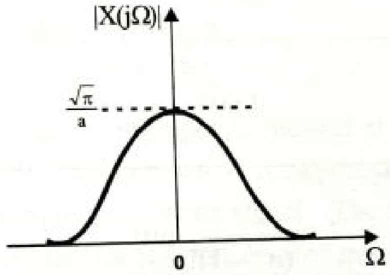
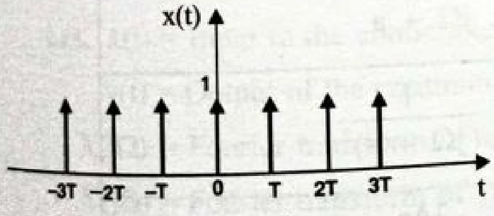
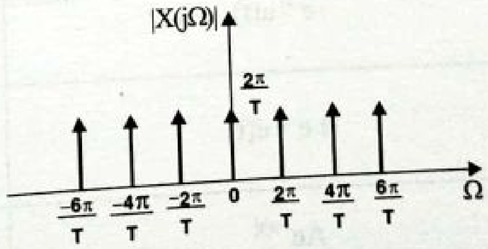
Table 4.4 : Continued.....

x(t)	X(jΩ) and Magnitude Spectrum
$x(t) = \frac{\Omega_0}{2\pi}\,\text{sinc}\left(\frac{\Omega_0}{2}t\right) = \frac{1}{\pi}\frac{\sin\left(\frac{\Omega_0}{2}t\right)}{t}$  Sinc pulse	$X(j\Omega) = \left[u\left(\Omega + \frac{\Omega_0}{2}\right) - u\left(\Omega - \frac{\Omega_0}{2}\right)\right]$  $\boxed{\Omega_0 = \frac{2\pi}{T}}$
$x(t) = e^{-a^2t^2}$  Gaussian pulse	$X(j\Omega) = \frac{\sqrt{\pi}}{a}e^{-\left(\frac{\Omega}{2a}\right)^2}$ 
$x(t) = \sum_{n=-\infty}^{+\infty}\delta(t - nT)$  Impulse train	$X(j\Omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty}\delta\left(\Omega - \frac{2\pi n}{T}\right)$ 

From table 4.4 the following observations are made.

1. The Fourier transform of a Gaussian pulse will be another Gaussian pulse.
2. The Fourier transform of an impulse train will be another impulse train.
3. The Fourier transform of a rectangular pulse will be a sinc pulse and viceversa.
4. The Fourier transform of a triangular pulse will be a squared sinc pulse.
5. The Fourier transform of a constant will be an impulse and vice-versa.

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Table 4.5 : Standard Fourier Tranform Pairs

$x(t)$	$X(j\Omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\Omega t_0}$
A where, A is constant	$2\pi A\delta(\Omega)$
$u(t)$	$\pi\delta(\Omega) + \frac{1}{j\Omega}$
$\mathrm{sgn}(t)$	$\frac{2}{j\Omega}$
$t\,u(t)$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{(j\Omega)^n}$
$t^n u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\frac{1}{(j\Omega + a)^2}$
$Ae^{-a\|t\|}$	$\frac{2Aa}{a^2 + \Omega^2}$
$Ae^{j\Omega_0 t}$	$2\pi A\,\delta(\Omega-\Omega_0)$
$\sin\Omega_0 t$	$\frac{\pi}{j}[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]$
$\cos\Omega_0 t$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$

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4.15 Solved Problems in Fourier Transform

Example 4.13

Determine the Fourier transform of following continuous time domain signals.

a) $x(t) = 1 - t^2$; for $|t| < 1$
$= 0$; for $|t| > 1$

b) $x(t) = e^{-at} \cos \Omega_0 t \; u(t)$

Solution

a) Given that, $x(t) = 1 - t^2$; for $|t| < 1$

$\therefore x(t) = 1 - t^2$; for $t = -1$ to $+1$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-1}^{+1} (1 - t^2)\, e^{-j\Omega t}\, dt = \int_{-1}^{+1} e^{-j\Omega t}\, dt - \int_{-1}^{+1} t^2 e^{-j\Omega t}\, dt$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[t^2 \frac{e^{-j\Omega t}}{-j\Omega} - \int 2t \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-1}^{1} \qquad \boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{j\Omega}\int t\, e^{-j\Omega t}\, dt\right]_{-1}^{1}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{j\Omega}\left(t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right)\right]_{-1}^{1} \qquad \boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{(j\Omega)^2}\left(-t\, e^{-j\Omega t} + \int e^{-j\Omega t}\, dt\right)\right]_{-1}^{1}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} - \frac{2}{\Omega^2}\left(-t\, e^{-j\Omega t} + \frac{e^{-j\Omega t}}{-j\Omega}\right)\right]_{-1}^{1}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2t\, e^{-j\Omega t}}{\Omega^2} + \frac{2e^{-j\Omega t}}{j\Omega^3}\right]_{-1}^{1}$$

$$= -\frac{e^{-j\Omega}}{j\Omega} + \frac{e^{j\Omega}}{j\Omega} - \left[-\frac{e^{-j\Omega}}{j\Omega} + \frac{2\, e^{-j\Omega}}{\Omega^2} + \frac{2\, e^{-j\Omega}}{j\Omega^3} + \frac{e^{j\Omega}}{j\Omega} + \frac{2\, e^{j\Omega}}{\Omega^2} - \frac{2e^{j\Omega}}{j\Omega^3}\right]$$

$$= -\frac{e^{-j\Omega}}{j\Omega} + \frac{e^{j\Omega}}{j\Omega} + \frac{e^{-j\Omega}}{j\Omega} - \frac{2\, e^{-j\Omega}}{\Omega^2} - \frac{2\, e^{-j\Omega}}{j\Omega^3} - \frac{e^{j\Omega}}{j\Omega} - \frac{2\, e^{j\Omega}}{\Omega^2} + \frac{2\, e^{j\Omega}}{j\Omega^3}$$

$$= -\frac{2}{\Omega^2}\left(e^{j\Omega} + e^{-j\Omega}\right) + \frac{2}{j\Omega^3}\left(e^{j\Omega} - e^{-j\Omega}\right) \qquad \boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}} \quad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= -\frac{2}{\Omega^2}\, 2\cos\Omega + \frac{2}{j\Omega^3}\, 2j\sin\Omega$$

$$= -\frac{4\cos\Omega}{\Omega^2} + \frac{4\sin\Omega}{\Omega^3}$$

$$= \frac{4}{\Omega^2}\left(\frac{\sin\Omega}{\Omega} - \cos\Omega\right)$$

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(b) Given that, $x(t) = e^{-at}\cos\Omega_0 t\, u(t)$

Since $u(t) = 1$, for $t \geq 0$, we can write,

$$x(t) = e^{-at}\cos\Omega_0 t \quad ; \quad \text{for } t \geq 0$$

By definition of Fourier transform,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_0^{\infty} e^{-at}\cos\Omega_0 t\, e^{-j\Omega t}\, dt = \int_0^{\infty} e^{-at}\left(\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\right) e^{-j\Omega t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-at} e^{j\Omega_0 t} e^{-j\Omega t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-at} e^{-j\Omega_0 t} e^{-j\Omega t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(a - j\Omega_0 + j\Omega)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-(a + j\Omega_0 + j\Omega)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(a - j\Omega_0 + j\Omega)t}}{-(a - j\Omega_0 + j\Omega)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-(a + j\Omega_0 + j\Omega)t}}{-(a + j\Omega_0 + j\Omega)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(a - j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a - j\Omega_0 + j\Omega)}\right] + \frac{1}{2}\left[\frac{e^{-\infty}}{-(a + j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a + j\Omega_0 + j\Omega)}\right]$$

$$= \frac{1}{2}\left[0 + \frac{1}{a - j\Omega_0 + j\Omega}\right] + \frac{1}{2}\left[0 + \frac{1}{a + j\Omega_0 + j\Omega}\right]$$

$e^{-\infty} = 0\ ;\ e^0 = 1$

$$= \frac{1}{2}\left[\frac{1}{(a + j\Omega) - j\Omega_0} + \frac{1}{(a + j\Omega) + j\Omega_0}\right]$$

$(a+b)(a-b) = a^2 - b^2$ | $j^2 = -1$

$$= \frac{1}{2}\left[\frac{(a + j\Omega) + j\Omega_0 + (a + j\Omega) - j\Omega_0}{(a + j\Omega)^2 + \Omega_0^2}\right]$$

$$= \frac{1}{2}\,\frac{2(a + j\Omega)}{(a + j\Omega)^2 + \Omega_0^2} = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$$

Example 4.14

Determine the Fourier transform of the rectangular pulse shown in fig 4.14.1.

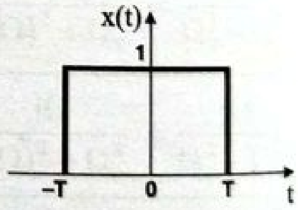

Fig 4.14.1.

Solution

The mathematical equation of the rectangular pulse is,

$$x(t) = 1 \quad ; \quad \text{for } t = -T \text{ to } +T$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{+T} 1 \times e^{-j\Omega t}\, dt = \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{+T}$$

$$= \frac{e^{-j\Omega T}}{-j\Omega} - \frac{e^{j\Omega T}}{-j\Omega} = \frac{1}{j\Omega}\left(e^{j\Omega T} - e^{-j\Omega T}\right) = \frac{1}{j\Omega}\, 2j\sin\Omega T$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= 2\,\frac{\sin\Omega T}{\Omega} = 2T\,\frac{\sin\Omega T}{\Omega T}$$

$$= 2T\,\mathrm{sinc}\,\Omega T$$

$$\frac{\sin\theta}{\theta} = \mathrm{sinc}\,\theta$$

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Example 4.15

Determine the Fourier transform of the triangular pulse shown in fig 4.15.1.

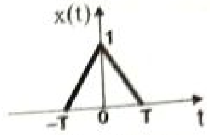

Fig 4.15.1.

Solution

The mathematical equation of triangular pulse is,

$$x(t) = 1 + \frac{t}{T} \quad ; \quad \text{for } t = -T \text{ to } 0$$
$$= 1 - \frac{t}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

(Please refer example 4.11 for the mathematical equation of triangular pulse).

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-j\Omega t}\, dt + \int_{0}^{T} \left(1 - \frac{t}{T}\right) e^{-j\Omega t}\, dt$$

$$= \int_{-T}^{0} e^{-j\Omega t}\, dt + \frac{1}{T}\int_{-T}^{0} t\, e^{-j\Omega t}\, dt + \int_{0}^{T} e^{-j\Omega t}\, dt - \frac{1}{T}\int_{0}^{T} t\, e^{-j\Omega t}\, dt$$

$$\boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} + \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-T}^{0} + \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T} - \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \int e^{-j\Omega t}\, dt\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \int e^{-j\Omega t}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{0} - e^{j\Omega T}\right] - \frac{1}{j\Omega T}\left[0 - \frac{e^{0}}{-j\Omega} + T\, e^{j\Omega T} + \frac{e^{j\Omega T}}{-j\Omega}\right] - \frac{1}{j\Omega}\left[e^{-j\Omega T} - e^{0}\right]$$
$$+ \frac{1}{j\Omega T}\left[T\, e^{-j\Omega T} - \frac{e^{-j\Omega T}}{-j\Omega} - 0 + \frac{e^{0}}{-j\Omega}\right]$$

$$= -\frac{1}{j\Omega} + \frac{e^{j\Omega T}}{j\Omega} - 0 + \frac{1}{T\Omega^2} - \frac{e^{j\Omega T}}{j\Omega} - \frac{e^{j\Omega T}}{T\Omega^2} - \frac{e^{-j\Omega T}}{j\Omega} + \frac{1}{j\Omega} + \frac{e^{-j\Omega T}}{j\Omega} - \frac{e^{-j\Omega T}}{T\Omega^2} - 0 + \frac{1}{T\Omega^2}$$

$$= \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\left(e^{j\Omega T} + e^{-j\Omega T}\right) = \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\, 2\cos\Omega T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right)$$

Alternatively the above result can be expressed as shown below.

$$\mathcal{F}\{x(t)\} = \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right) = \frac{2}{T\Omega^2}\left(1 - \cos 2\left(\frac{\Omega T}{2}\right)\right)$$

$$= \frac{2}{T\Omega^2}\left(2\sin^2\frac{\Omega T}{2}\right) = T\,\frac{4}{T^2\Omega^2}\sin^2\frac{\Omega T}{2} = T\,\frac{\sin^2\left(\frac{\Omega T}{2}\right)}{\left(\frac{\Omega T}{2}\right)^2}$$

$$\boxed{\sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$= T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2 = T\left(\operatorname{sinc}\frac{\Omega T}{2}\right)^2$$

$$\boxed{\frac{\sin\theta}{\theta} = \operatorname{sinc}\theta}$$

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Example 4.16

Determine the inverse Fourier transform of the following functions, using partial fraction expansion technique.

a) $X(j\Omega) = \dfrac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12}$ b) $X(j\Omega) = \dfrac{j\Omega + 7}{(j\Omega + 3)^2}$

Solution

a) Given that, $X(j\Omega) = \dfrac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12} = \dfrac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)}$

By partial fraction expansion technique we can write,

$$X(j\Omega) = \frac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)} = \frac{k_1}{j\Omega + 3} + \frac{k_2}{j\Omega + 4}$$

$$k_1 = \frac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)} \times (j\Omega + 3)\Bigg|_{j\Omega = -3} = \frac{3(-3) + 14}{-3 + 4} = 5$$

$$k_2 = \frac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)} \times (j\Omega + 4)\Bigg|_{j\Omega = -4} = \frac{3(-4) + 14}{-4 + 3} = -2$$

$$\therefore X(j\Omega) = \frac{5}{j\Omega + 3} - \frac{2}{j\Omega + 4} \quad .....(1)$$

We know that, $\mathcal{F}\{e^{-at}\,u(t)\} = \dfrac{1}{j\Omega + a}$ .....(2)

Using equation (2), the inverse Fourier transform of equation (1) is,

$$x(t) = 5\,e^{-3t}\,u(t) - 2\,e^{-4t}\,u(t)$$

---

b) Given that, $X(j\Omega) = \dfrac{j\Omega + 7}{(j\Omega + 3)^2}$

By partial fraction expansion technique $X(j\Omega)$ can be written as,

$$\therefore X(j\Omega) = \frac{k_1}{(j\Omega + 3)^2} + \frac{k_2}{j\Omega + 3}$$

$$k_1 = \frac{j\Omega + 7}{(j\Omega + 3)^2} \times (j\Omega + 3)^2\Bigg|_{j\Omega = -3} = -3 + 7 = 4$$

$$k_2 = \frac{d}{d(j\Omega)}\left[\frac{j\Omega + 7}{(j\Omega + 3)^2} \times (j\Omega + 3)^2\right]\Bigg|_{j\Omega = -3} = \frac{d}{d(j\Omega)}[j\Omega + 7]\Bigg|_{j\Omega = -3} = 1$$

$$\therefore X(j\Omega) = \frac{4}{(j\Omega + 3)^2} + \frac{1}{j\Omega + 3} \quad .....(3)$$

We know that, $\mathcal{F}\{e^{-at}\,u(t)\} = \dfrac{1}{j\Omega + a}$ .....(4)

$\mathcal{F}\{t\,e^{-at}\,u(t)\} = \dfrac{1}{(j\Omega + a)^2}$ .....(5)

Using equations (4) and (5), the inverse Fourier transform of equation (3) is,

$$x(t) = 4t\,e^{-3t}\,u(t) + e^{-3t}\,u(t) = (4t + 1)\,e^{-3t}\,u(t)$$

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4.12 Fourier Transform of a Periodic Signal

Let, $x(t)$ = Continuous time periodic signal

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \text{Fourier transform of } x(t)$$

The exponential form of Fourier series representation of $x(t)$ is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Omega_0 t}$$

From equation (4.9)

On taking Fourier transform of the above equation we get,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} c_n \, \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{+\infty} c_n \, 2\pi \, \delta(\Omega - n\Omega_0) = 2\pi \sum_{n=-\infty}^{+\infty} c_n \, 2\pi \, \delta(\Omega - n\Omega_0)$$

Using equation (4.61)

$$= \ldots\ldots + 2\pi c_{-3}\, \delta(\Omega + 3\Omega_0) + 2\pi c_{-2}\, \delta(\Omega + 2\Omega_0) + 2\pi c_{-1}\, \delta(\Omega + \Omega_0) + 2\pi c_0\, \delta(\Omega) + 2\pi c_1\, \delta(\Omega - \Omega_0) + 2\pi c_2\, \delta(\Omega - 2\Omega_0) + 2\pi\, c_3\, \delta(\Omega - 3\Omega_0) + \ldots\ldots \quad \ldots\ldots(4.65)$$

The magnitude of each term of equation (4.65) represents an impulse, located at its harmonic frequency in the magnitude spectrum. Hence we can say that the Fourier transform of a periodic continuous time signal consists of impulses located at the harmonic frequencies of the signal. The magnitude of each impulse is 2π times the magnitude of Fourier coefficient, i.e., the magnitude of n^{th} impulse is $2\pi\, |c_n|$.

4.13. Analysis of LTI Continuous Time System Using Fourier Transform

4.13.1 Transfer Function of LTI Continuous Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI continuous time system in frequency domain.

Let, $x(t)$ = Input to the continuous time system

$y(t)$ = Output of the continuous time system

$\therefore X(j\Omega)$ = Fourier transform of $x(t)$

$Y(j\Omega)$ = Fourier transform of $y(t)$

$$\text{Now, Transfer function} = \frac{Y(j\Omega)}{X(j\Omega)} \quad \ldots\ldots(4.66)$$

The transfer function of LTI continuous time system in frequency domain can be obtained from the differential equation governing the input-output relation of an LTI continuous time system, (refer chapter-2, equation (2.13)),

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + \ldots\ldots + a_{N-1}\frac{d}{dt}y(t) + a_N\, y(t) = b_0\frac{d^M}{dt^M}x(t) + b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + \ldots\ldots + b_{M-1}\frac{d}{dt}x(t) + b_M x(t)$$

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On taking Fourier transform of the above equation and rearranging the resultant equation as a ratio of $Y(j\Omega)$ and $X(j\Omega)$, the transfer function of LTI continuous time system in frequency domain is obtained.

Impulse Response and Transfer Function

Consider an LTI continuous time system, $\mathcal{H}$ shown in fig 4.38. Let x(t) and y(t) be the input and output of the system respectively.

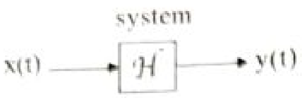

Fig 4.38.

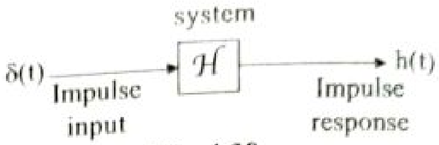

Fig 4.39.

For a continuous time system $\mathcal{H}$, if the input is impulse signal $\delta(t)$ as shown in fig 4.39, then the output is called ***impulse response***, which is denoted by h(t).

The importance of impulse response is that the response for any input to LTI system is given by convolution of input and impulse response.

Symbolically, the convolution operation is denoted as,

$$y(t) = x(t) * h(t) \qquad .....(4.67)$$

where, " $*$ " is the symbol for convolution.

Mathematically, the convolution operation is defined as,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

where, τ is the dummy variable for integration.

Let, $H(j\Omega)$ = Fourier transform of h(t)

$X(j\Omega)$ = Fourier transform of x(t)

$Y(j\Omega)$ = Fourier transform of y(t)

Now, by convolution property of Fourier transform we get,

$$\mathcal{F}\{x(t) * h(t)\} = X(j\Omega)\, H(j\Omega) \qquad .....(4.68)$$

Using equation (4.67), the equation (4.68) can be written as,

$$\mathcal{F}\{y(t)\} = X(j\Omega)\, H(j\Omega)$$

$$\therefore\ Y(j\Omega) = X(j\Omega)\, H(j\Omega)$$

$$\therefore\ H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} \qquad .....(4.69)$$

From equations (4.66) and (4.69) we can say that the ***transfer function in frequency domain*** is given by Fourier transform of impulse response.

$$\boxed{\therefore\ H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}} \qquad .....(4.70)$$

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4.13.2 Response of LTI Continuous Time System Using Fourier Transform

Consider the transfer function of LTI continuous time system, $H(j\Omega)$.

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

Now, response in frequency domain, $Y(j\Omega) = H(j\Omega)\, X(j\Omega)$

The response function $Y(j\Omega)$ will be a rational function of $j\Omega$, and so $Y(j\Omega)$ can be expressed as a ratio of two factorized polynomial in $j\Omega$ as shown below.

$$Y(j\Omega) = \frac{(j\Omega + z_1)(j\Omega + z_2)(j\Omega + z_3)\,.....}{(j\Omega + p_1)(j\Omega + p_2)(j\Omega + p_3)\,.....} \quad .....(4.71)$$

By partial fraction expansion technique the equation (4.71) can be expressed as shown below.

$$Y(j\Omega) = \frac{k_1}{j\Omega + p_1} + \frac{k_2}{j\Omega + p_2} + \frac{k_3}{j\Omega + p_3} + \,..... \quad .....(4.72)$$

where, k_1, k_2, k_3, ... are residues.

Now the time domain response y(t) can be obtained by taking inverse Fourier transform of equation (4.72). The inverse Fourier transform of each term in equation (4.72) can be obtained by comparing the terms with standard Fourier transform pair listed in table 4.5.

From table - 4.5, we get, $\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{a + j\Omega}$ .....(4.73)

Using equation (4.73), the inverse Fourier transform of equation (4.72) can be obtained as shown below.

$$y(t) = k_1\, e^{-p_1 t}\, u(t) + k_2\, e^{-p_2 t}\, u(t) + k_3\, e^{-p_3 t}\, u(t) + \,..... \quad .....(4.74)$$

Since the transfer function is defined with zero initial conditions, the response obtained by using equation (4.74) is the time domain steady state (or forced) response of the LTI continuous time system.

Note: *Only steady state or forced response alone can be computed via frequency domain*

4.13.3 Frequency Response of LTI Continuous Time System

The output y(t) of an LTI continuous time system is given by convolution of h(t) and x(t).

$$\therefore\; y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t - \tau)\, d\tau \quad .....(4.75)$$

Consider a special class of input (sinusoidal input), $\boxed{Ae^{j\Omega t} = A(\cos\Omega t + j\sin\Omega t)}$

$$x(t) = A\, e^{j\Omega t} \quad .....(4.76)$$

where, A = Amplitude ; Ω = Angular frequency in rad/sec

$$\therefore\; x(t - \tau) = Ae^{j\Omega(t - \tau)} \quad .....(4.77)$$

On substituting for $x(t - \tau)$ from equation (4.77) in equation (4.75) we get,

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, Ae^{j\Omega(t - \tau)}\, d\tau$$

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$$\therefore y(t) = \int_{-\infty}^{+\infty} h(\tau)\, Ae^{j\Omega t}\, e^{-j\Omega\tau} d\tau$$

$$= Ae^{j\Omega t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega\tau} d\tau \qquad .....(4.78)$$

By the definition of Fourier transform,

Replace t by τ.

$$H(j\Omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t} dt = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega\tau} d\tau \qquad .....(4.79)$$

Using equations (4.76) and (4.79), the equation (4.78) can be written as,

$$y(t) = x(t)\; H(j\Omega) \qquad .....(4.80)$$

From equation (4.80) we can say that if a complex sinusoidal signal is given as input signal to an LTI continuous time system, then the output is also a sinusoidal signal of the same frequency modified by H(jΩ). Hence H(jΩ) is called the ***frequency response*** of the continuous time system.

Since the H(jΩ) is a complex function of Ω, the multiplication of H(jΩ) with input produces a change in the amplitude and phase of the input signal, and the modified input signal is the output signal. Therefore, an LTI system is characterized in the frequency domain by its frequency response.

The function H(jΩ) is a complex quantity and so it can be expressed as magnitude function and phase function.

$$\therefore H(j\Omega) = |H(j\Omega)| \;\angle H(j\Omega)$$

where, $|H(j\Omega)|$ = Magnitude function

$\angle H(j\Omega)$ = Phase function

The sketch of magnitude function and phase function with respect to Ω will give the frequency response graphically.

Let, $H(j\Omega) = H_r(j\Omega) + jH_i(j\Omega)$

where, $H_r(j\Omega)$ = Real part of $H(j\Omega)$

$H_i(j\Omega)$ = Imaginary part of $H(j\Omega)$

The ***magnitude function*** is defined as,

$$|H(j\Omega)|^2 = H(j\Omega)\, H^*(j\Omega) = [H_r(j\Omega) + jH_i(j\Omega)]\,[H_r(j\Omega) - jH_i(j\Omega)]$$

where, $H^*(j\Omega)$ is complex conjugate of $H(j\Omega)$

$$\therefore |H(j\Omega)|^2 = H_r^2(j\Omega) + H_i^2(j\Omega) \;\Rightarrow\; |H(j\Omega)| = \sqrt{H_r^2(j\Omega) + H_i^2(j\Omega)}$$

The ***phase function*** is defined as,

$$\angle H(j\Omega) = \text{Arg}[H(j\Omega)] = \tan^{-1}\left[\frac{H_i(j\Omega)}{H_r(j\Omega)}\right]$$

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From equation (4.70) we can say that the frequency response $H(j\Omega)$ of an LTI continuous time system is same as transfer function in frequency domain and so, the frequency response is also given by the ratio of Fourier transform of output to Fourier transform of input.

$$\therefore \text{ Frequency response, } H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} \quad .....(4.81)$$

Advantages of frequency response analysis

1. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments .
2. The transfer function of complicated systems can be determined experimentally by frequency response tests.
3. The design and parameter adjustment is carried out more easily in frequency domain.
4. In frequency domain, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.
5. The frequency response analysis and designs can be extended to certain nonlinear systems.

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Example 4.17

Determine the convolution of $x_1(t) = e^{-2t}\,u(t)$ and $x_2(t) = e^{-6t}\,u(t)$, using Fourier transform.

Solution

Let, $X_1(j\Omega)$ = Fourier transform of $x_1(t)$

$X_2(j\Omega)$ = Fourier transform of $x_2(t)$

By convolution property of Fourier transform,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\Omega)\,X_2(j\Omega)$$

Let,
$$X(j\Omega) = X_1(j\Omega)\,X_2(j\Omega)$$
$$= \mathcal{F}\{e^{-2t}\,u(t)\} \times \mathcal{F}\{e^{-6t}\,u(t)\}$$
$$= \frac{1}{j\Omega + 2} \times \frac{1}{j\Omega + 6}$$

By partial fraction expansion technique $X(j\Omega)$ can be expressed as,

$$X(j\Omega) = \frac{1}{(j\Omega + 2)(j\Omega + 6)} = \frac{k_1}{j\Omega + 2} + \frac{k_2}{j\Omega + 6}$$

$$k_1 = \frac{1}{(j\Omega + 2)(j\Omega + 6)} \times (j\Omega + 2)\bigg|_{j\Omega = -2} = \frac{1}{-2 + 6} = \frac{1}{4} = 0.25$$

$$k_2 = \frac{1}{(j\Omega + 2)(j\Omega + 6)} \times (j\Omega + 6)\bigg|_{j\Omega = -6} = \frac{1}{-6 + 2} = -\frac{1}{4} = -0.25$$

$$\therefore\ X(j\Omega) = \frac{0.25}{j\Omega + 2} - \frac{0.25}{j\Omega + 6}$$

On taking inverse Fourier transform of the above equation we get,

$$\boxed{\mathcal{F}\{e^{-at}\,u(t)\} = \frac{1}{j\Omega + a}}$$

$$x(t) = 0.25\,e^{-2t}\,u(t) - 0.25\,e^{-6t}\,u(t)$$
$$= 0.25(e^{-2t} - e^{-6t})\,u(t)$$

Example 4.18

The impulse response of an LTI system is $h(t) = 2\,e^{-3t}u(t)$.

Find the response of the system for the input $x(t) = 2e^{-5t}u(t)$, using Fourier transform.

Solution

Given that, $x(t) = 2\,e^{-5t}\,u(t)$. .....(1)

$$\therefore\ X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{2\,e^{-5t}\,u(t)\} = \frac{2}{j\Omega + 5}$$

$$\boxed{\mathcal{F}\{e^{-at}\,u(t)\} = \frac{1}{j\Omega + a}}$$

Given that, $h(t) = 2\,e^{-3t}u(t)$. .....(2)

$$\therefore\ H(j\Omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\{2\,e^{-3t}\,u(t)\} = \frac{2}{j\Omega + 3} \quad .....(3)$$

For LTI system, the response, y(t) = = x(t) * h(t)

On taking Fourier transform of equation (3) we get,

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\}$$

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Let, $\mathcal{F}\{y(t)\} = Y(j\Omega)$.

$$\therefore Y(j\Omega) = \mathcal{F}\{x(t) * h(t)\}$$

$$= X(j\Omega)\,H(j\Omega)$$ Using convolution property of Fourier transform.

$$= \frac{2}{j\Omega + 5} \times \frac{2}{j\Omega + 3} = \frac{4}{(j\Omega + 5)(j\Omega + 3)}$$ Using equations (1) and (2)

By partial fraction expansion technique, the above equation can be written as,

$$Y(j\Omega) = \frac{4}{(j\Omega + 5)(j\Omega + 3)} = \frac{k_1}{j\Omega + 5} + \frac{k_2}{j\Omega + 3}$$

$$k_1 = \frac{4}{(j\Omega + 5)(j\Omega + 3)} \times (j\Omega + 5)\Big|_{j\Omega = -5} = \frac{4}{-5 + 3} = -2$$

$$k_2 = \frac{4}{(j\Omega + 5)(j\Omega + 3)} \times (j\Omega + 3)\Big|_{j\Omega = -3} = \frac{4}{-3 + 5} = 2$$

$$\therefore Y(j\Omega) = -\frac{2}{j\Omega + 5} + \frac{2}{j\Omega + 3}$$

On taking inverse Fourier transform of $Y(j\Omega)$ we get $y(t)$.

$$y(t) = \mathcal{F}^{-1}\{Y(j\Omega)\} = \mathcal{F}^{-1}\left\{-\frac{2}{j\Omega + 5} + \frac{2}{j\Omega + 3}\right\}$$

$$= -2\,e^{-5t}\,u(t) + 2\,e^{-3t}\,u(t) = 2\left(e^{-3t} - e^{-5t}\right)u(t)$$

Example 4.19

Determine the Fourier transform of the periodic pulse function shown in fig 4.19.1.

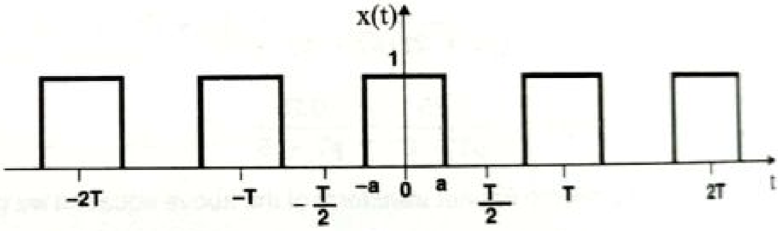

Fig 4.19.1.

Solution

The mathematical equation for one period of the periodic pulse function is,

$$x(t) = 1 \;;\; t = -a \text{ to } +a$$

$$= 0 \;;\; t = -\frac{T}{2} \text{ to } -a \text{ and } t = a \text{ to } \frac{T}{2}$$

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,e^{-jn\Omega_0 t}\,dt = \frac{1}{T}\int_{-a}^{+a} e^{-jn\Omega_0 t}\,dt = \frac{1}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-a}^{+a} = \frac{1}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-a}^{+a}$$ $\Omega_0 = \frac{2\pi}{T}$

$$= \frac{1}{T}\left[\frac{e^{-jn\Omega_0 a}}{-jn\Omega_0} - \frac{e^{jn\Omega_0 a}}{-jn\Omega_0}\right] = \frac{1}{T}\,\frac{2}{n\Omega_0}\left[\frac{e^{jn\Omega_0 a} - e^{-jn\Omega_0 a}}{2j}\right]$$ $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \frac{1}{T}\,\frac{2}{n}\,\frac{T}{2\pi}\,\sin(n\Omega_0 a) = \frac{1}{n\pi}\,\sin(an\Omega_0)$$ ....(1)

The exponential Fourier series representation of the periodic pulse function is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\,e^{jn\Omega_0 t}$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n\,e^{jn\Omega_0 t}\right\}$$

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$$\therefore X(j\Omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} c_n \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{+\infty} c_n 2\pi \delta(\Omega - n\Omega_0)$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{n\pi} \sin(an\Omega_0) 2\pi \delta(\Omega - n\Omega_0)$$

$$= \sum_{n=-\infty}^{+\infty} \frac{2\sin(an\Omega_0)}{n} \delta(\Omega - n\Omega_0)$$

$$= \sum_{n=-\infty}^{+\infty} 2a\Omega_0 \left(\frac{\sin an\Omega_0}{an\Omega_0}\right) \delta(\Omega - n\Omega_0) = \sum_{n=-\infty}^{+\infty} 2a\Omega_0 \,\mathrm{sinc}(an\Omega_0) \,\delta(\Omega - n\Omega_0)$$

$\mathcal{F}\{x(t)\} = X(j\Omega)$
$\mathcal{F}\left\{e^{jn\Omega_0 t}\right\} = 2\pi\delta(\Omega - n\Omega_0)$
Substituting for c_n from equation (1).
$\frac{\sin\theta}{\theta} = \mathrm{sinc}\,\theta$

Example 4.20

Determine the Fourier transform of the periodic impulse function shown in fig 4.20.1.

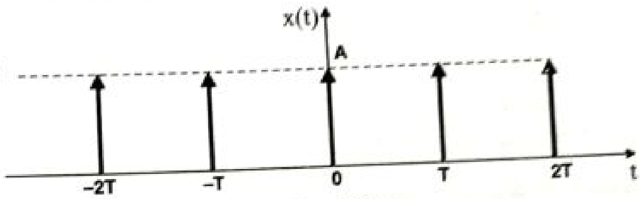

Fig 4.20.1.

Solution

The mathematical equation for one period of the periodic impulse function is,

$$x(t) = A\,\delta(t) \quad ; \quad \text{for } t = -\frac{T}{2} \text{ to } +\frac{T}{2}$$

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{+T/2} A\,\delta(t)\, e^{-jn\Omega_0 t}\, dt = \frac{A}{T} e^{-jn\Omega_0 t}\Big|_{t=0} = \frac{A}{T} \quad \text{....(1)}$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

The Exponential Fourier series representation of the periodic impulse train is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}\right\}$$

$$\therefore X(j\Omega) = \sum_{n=-\infty}^{+\infty} c_n \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{+\infty} c_n\, 2\pi\,\delta(\Omega - n\Omega_0)$$

$$= \sum_{n=-\infty}^{+\infty} \frac{A}{T}\, 2\pi\,\delta(\Omega - n\Omega_0) = \sum_{n=-\infty}^{+\infty} A\Omega_0\, \delta(\Omega - n\Omega_0)$$

$\mathcal{F}\{x(t)\} = X(j\Omega)$
$\mathcal{F}\left\{e^{jn\Omega_0 t}\right\} = 2\pi\delta(\Omega - n\Omega_0)$
On substituting for c_n from equation (1)

The magnitude spectrum of $X(j\Omega)$ is shown in fig 1, which is also a periodic impulse function of Ω.

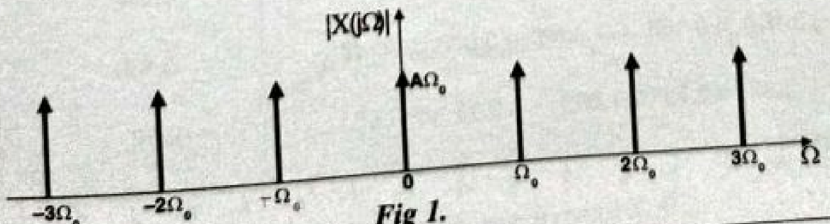

Fig 1.

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