Heisenberg Uncertainty Principle

Heisenberg uncertainty principle states that two canonically conjugate quantities (variables) is incompatible i.e. cannot be measured exactly simultaneously.

One of the most novel concept in wave mechanics is the uncertainty. It is known that commutation operation of certain pairs of operators is not zero. This implies that there is mutual disturbance between the observables corresponding to these two operators and hence they cannot be measured simultaneously.

The fundamental question in quantum mechanics is "What is a particle?" in classical mechanics, a particle is a point object having precise position and momentum at every instant of time and is described by a harmonic wave which extends over all space. Hence it is impossible to get any idea regarding the position of the particle. In quantum mechanics; such a picture for a particle is not valid; at least in the atomic domain.

Actually the motion of a particle is surrounded by a wave packet (group of waves) i.e. a wave which is confined within a small region, the corresponding wave function has a finite amplitude and wavelength. The particle may be found anywhere within the range of wavelength. This implies that the position of the particle is indeterminate within this range. The uncertainty in determining the position of the particle may be decreased by the construction of a localized wave packet in space. However, this wave packet will spread in space with time and ultimately uncertainty creeps in.

Moreover, the wave packet has a velocity spread and hence there is uncertainty about the velocity or momentum of the particle. For a large wave packet with many crests, the velocity spread is very small so that the particle velocity can be determined very accurately but the position is very uncertain. On the other hand, for a small wave packet, the position of the particle can be determined accurately but as the velocity spread is large it becomes very uncertain.

Now, whether this uncertainty condition corresponds to any physical reality? This question gives rise to two possibilities:

1. *The motions of particles are completely defined with certainty so that they cannot be represented by wave packets and wave mechanics fail.*

2. *The motions of the particles can be represented by wave packets so that it is impossible to measure simultaneously certain pairs of dynamical variables without any uncertainty.*

Experimental evidences for the wave nature of matter confirms the built of second possibility. All experiments on microscopic objects yield results which are consistent with the uncertainty principle. Actually, uncertainty principle protects wave particle duality.

PROOF:

1. Position-Momentum uncertainty relation (Thought experiment by γ-ray microscope)

This thought experiment was first proposed by Heisenberg. Generally we use a microscope of high resolving power to observe the position of a small object like an electron. As resolving power is inversely proportional to the wavelength, we must use light rays of shortest wavelength such as γ-rays to minimize the uncertainty.

The image of the electron formed by the microscope will be a diffraction pattern consisting of a central bright disc surrounded by alternate dark and bright rings. As the electron may be anywhere within the central bright disc, the uncertainty in the position of the electron is the diameter (Δx) of the central bright disc.

Therefore, the uncertainty in the position of the electron = Δx

Now, according to the Rayleigh's criterion of resolution, the resolving power (R) of an optical instrument is the distance of the first minimum from the central maximum of the diffraction pattern.

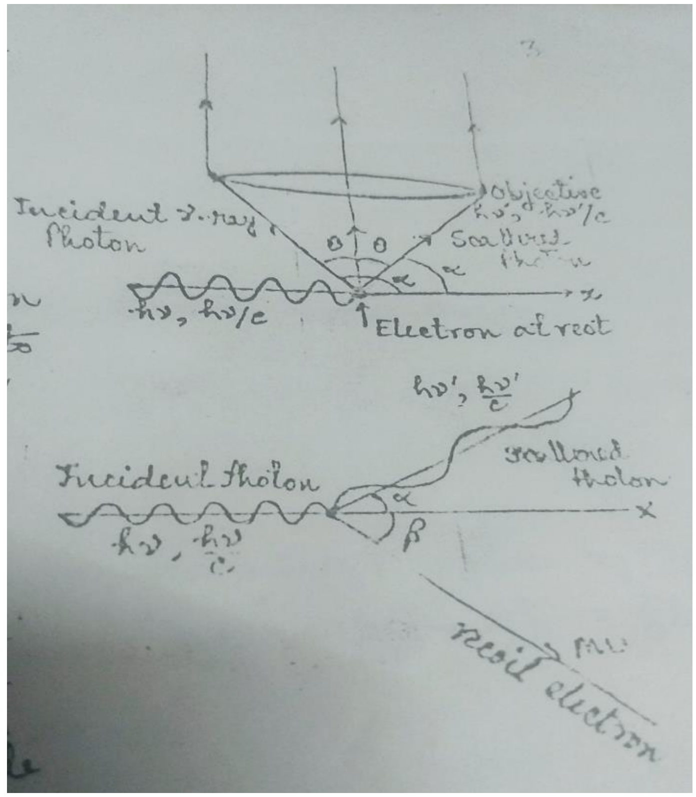
Therefore, $R = \frac{\lambda}{2Sin\theta}$; where λ = wavelength of light used for illuminating the object.

θ = half the angle subtended by the objective lens of the microscope at the position of the object under investigation.

But in this case, $R = \frac{\Delta x}{2}$

So, $\frac{\Delta x}{2} = \frac{\lambda}{2Sin\theta}$ i.e. $\boldsymbol{\Delta x = \frac{\lambda}{Sin\theta}}$

In order to observe where the electron is, at least one photon must be scattered into the microscope after collision with the electron at rest. In this scattering process, the photon transfers momentum to the electron according to the rules of Compton effect as illustrated in the adjoining diagram.



The scattered photon must travel within the cone of semi-vertical angle θ i.e. scattering angle α can extent from (90 – θ) to (90 + θ). According to the principle of conservation of momentum along x direction, we have:

$$\frac{h\vartheta}{c} = \frac{h\vartheta'}{c}cos\alpha + mvcos\beta$$

Therefore, the component of momentum of the electron imparted by the photon along x-direction is:

$p_x = mvcos\beta = \frac{h\vartheta}{c} - \frac{h\vartheta'}{c}cos\alpha = \frac{h}{c}(\vartheta - \vartheta' cos\alpha)$; *[α = (90 – θ), so Cosα = Cos(90 - θ) = Sinθ*

$p_x = \frac{h}{c}(\vartheta - \vartheta' sin\theta)$*]*

So the x-component of momentum can extent from $\frac{h}{c}(\vartheta - \vartheta' sin\theta)$ *to* $\frac{h}{c}(\vartheta + \vartheta' sin\theta)$. *So the uncertainty in momentum is* $\pm\frac{h}{c}\vartheta' sin\theta$ *i.e.* $\pm\frac{h}{\lambda}sin\theta$.

$$\boldsymbol{\Delta p_x = \frac{h}{\lambda}sin\theta}$$

So the average uncertainty between position and momentum are inter-related by the relation

$$\Delta x.\Delta p_x = \frac{\lambda}{sin\theta}.\frac{h}{\lambda}sin\theta = h$$

So in general $\boldsymbol{\Delta x.\Delta p_x} \geq \hbar$

2. Time-Energy Uncertainty Relation

The motion of a particle is associated with a wave packet which moves with the group velocity v, as the uncertainty in position is Δx, the instant at which it crosses a given point in which path carries an uncertainty Δt is given by, $\Delta t = \frac{\Delta x}{v}$

If Δp is the uncertainty in momentum, then

$$\Delta p \,.\, \Delta x = h \;\; i.e. \Delta x = \frac{h}{\Delta p}$$

$$So,\ \Delta t = \frac{h}{v\Delta p}$$

If m be the mass of the particle then its kinetic energy E is given by,

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

$$so, \frac{\partial E}{\partial p} = \frac{p}{m} = v$$

Now, the uncertainty in the measurement of E is given by,

$$\Delta E = \frac{\partial E}{\partial p} \,.\, \Delta p = v\Delta p$$

$$so, \Delta t = \frac{h}{\Delta E} \;\; i.e. \Delta E. \Delta t = h$$

So, in general we can write, $\Delta \boldsymbol{E} \,.\, \Delta \boldsymbol{t} \geq \hbar$.

Uncertainty comes due to the quantum properties of the measuring agent (light). As long as all possible agents (matter and light) have quantum properties no measurement can even in principle, lead to precise determination of two conjugate variables.