Implicit Functions and Their Derivatives

So far, we have been working only with functions in which the endogenous or dependent variables are explicit functions of the exogenous or independent variables. In other words, all the functions we have studied have had the x_i's on the right side and the y on the left side:

$$y = F(x_1, \ldots, x_n). \tag{1}$$

When the variables are separated as in (1), we say that the endogenous variable is an **explicit function** of the exogenous variables.

This ideal situation does not always occur in economic models. Frequently, the equations which arise naturally, for example, as first order conditions in a maximization problem, have the exogenous variables mixed in with the endogenous variables, as in

$$G(x_1, x_2, \ldots, x_n, y) = 0. \tag{2}$$

If for each $(x_1, \ldots, x_n)$ equation (2) determines a corresponding value y, we say that the equation (2) defines the endogenous variable y as an **implicit function** of the exogenous variables $x_1, \ldots, x_n$. An expression like (2) is often so complicated that one cannot solve it to separate the exogenous variables on one side and the endogenous on the other, as in (1). However, we still want to answer the basic question: how does a small change in one of the exogenous variables affect the value of the endogenous variable? This chapter will demonstrate how to answer this question for implicit functions.

15.1 IMPLICIT FUNCTIONS

Examples

Let's start with some simple examples.

Example 15.1 The equations

$$4x + 2y = 5 \quad \text{or} \quad 4x + 2y - 5 = 0 \tag{3}$$

express y as an implicit function of x. Of course, in this case, we can easily solve (3) and write y as an explicit function of x:

$$y = 2.5 - 2x.$$

Example 15.2 A more complex example of an implicit function is the equation

$$y^2 - 5xy + 4x^2 = 0. \tag{4}$$

We substitute any specified value of x into (4) and then solve the resulting quadratic equation for y. For example, when $x = 0$, (4) becomes $y^2 = 0$, whose solution is $y = 0$. When $x = 1$, (4) becomes $y^2 - 5y + 4 = 0$, whose solutions are $y = 1$ and $y = 4$. (When there are more than one choice of y for a given value of x, there is often some additional information which leads to a choice of a single y value.) Even though (4) is more complex than (3), we can still convert (4) into an explicit function (actually, a correspondence) by applying the quadratic formula to it:

$$y = \frac{5x \pm \sqrt{25x^2 - 16x^2}}{2} = \frac{1}{2}(5x \pm 3x) = \begin{cases} 4x \\ x. \end{cases}$$

Example 15.3 Applying the quadratic formula to the *implicit* function: $xy^2 - 3y - e^x = 0$ yields an *explicit* function

$$y = \frac{1}{2x}(3 \pm \sqrt{9 + 4xe^x}).$$

However, this explicit function may very well be more difficult to work with than the original implicit function.

Example 15.4 Changing one exponent in (4) to construct the implicit function

$$y^5 - 5xy + 4x^2 = 0 \tag{5}$$

yields an expression which cannot be solved into an explicit function because there is no general formula for solving quintic equations. However, (5) still defines y as a function of x. For example, when $x = 0$, (5) becomes $y^5 = 0$, whose solution is $y = 0$. When $x = 1$, (5) becomes $y^5 - 5y + 4 = 0$, with solution $y = 1$.

Example 15.5 Consider a profit-maximizing firm that uses a single input x at a cost of w dollars per unit to produce a single output via a production function $y = f(x)$. If output sells for p dollars a unit, the firm's profit function for any fixed p and w is

$$\Pi(x) = p \cdot f(x) - w \cdot x.$$

One takes the x-derivative of this profit function to derive the equation for the profit-maximizing choice of x:

$$pf'(x) - w = 0. \tag{6}$$

Think of p and w as exogenous variables. For each choice of p and w, the firm will want to choose x that satisfies (6). There is no reason to limit the models to production functions for which (6) can be solved explicitly for x in terms of p and w. To study the profit-maximizing behavior of a general firm, we need to work with (6) as defining x as an *implicit* function of p and w. We will want to know, for example, how the optimal choice of input x changes as p or w increases. If there are multiple solutions x of (6) for a given p and w, we can usually choose among the solution candidates by using second order conditions for a maximum or by looking for the *global* maximizer.

The fact that we can write down an implicit function $G(x, y) = c$ does not mean that this equation automatically defines y as a function of x. For example, consider the simple implicit function

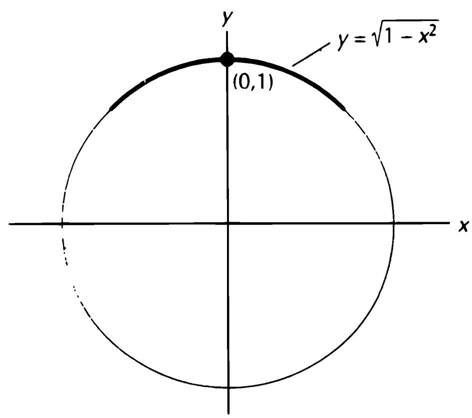
$$x^2 + y^2 = 1. \tag{7}$$

When $x > 1$, there is no y which satisfies (7). However, usually we start with a specific solution (x_0, y_0) of the implicit equation $G(x, y) = c$ and ask if we vary x a little from x_0, can we find a y near the original y_0 that satisfies the equation. For example, if we start with the solution $x = 0, y = 1$ of (7) and vary x a little, we can find a unique $y = \sqrt{1 - x^2}$ near $y = 1$ that corresponds to the new x. We can even draw the graph of this explicit relationship around the point (0, 1), as we do in Figure 15.1.

However, if we start at the solution $x = 1, y = 0$ of (7), then no such functional relationship exists. As Figure 15.2 indicates, if we increase x a little to $x = 1 + \varepsilon$, then there is no corresponding y so that $(1 + \varepsilon, y)$ solves (7). If we decrease x a little to $1 - \varepsilon$, then there are two equally good candidates for y *near* $y = 0$, namely

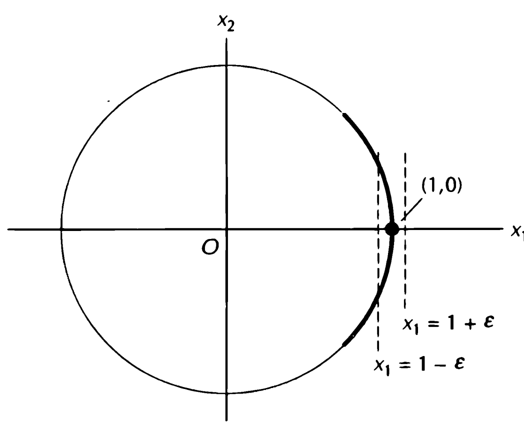
$$y = +\sqrt{2\varepsilon - \varepsilon^2} \quad \text{and} \quad y = -\sqrt{2\varepsilon - \varepsilon^2}.$$

As Figure 15.2 illustrates, because the curve $x^2 + y^2 = 1$ is vertical around (1, 0), it does *not* define y as a function of x there.



The graph of $x^2 + y^2 = 1$ *near the point* (0, 1).

Figure 15.1



The graph of $x^2 + y^2 = 1$ *near the point* (1, 0).

Figure 15.2

The Implicit Function Theorem for $\mathbf{R}^2$

For a given implicit function $G(x, y) = c$ and a specified solution point (x_0, y_0), we want to know the answers to the following two questions:

(1) Does $G(x, y) = c$ determine y as a continuous function of x for x near x_0 and y near y_0?
(2) If so, how do changes in x affect the corresponding y's?

Let's phrase these two questions more analytically.

(1) Given the implicit equation $G(x, y) = c$ and a point (x_0, y_0) such that $G(x_0, y_0) = c$, does there exist a continuous function $y = y(x)$ defined on an interval I about x_0 so that:
   (a) $G(x, y(x)) = c$ for all x in I and
   (b) $y(x_0) = y_0$?
(2) If $y(x)$ exists and is differentiable, what is $y'(x_0)$?

Notice that the statement "$y(x)$ exists" is much more general than the statement "an explicit function $y(x)$ can be written down."

It turns out that the answers of these two questions are closely related to each other in that if the first question has a positive answer, one can easily use the Chain Rule to compute a formula for $y'(x)$ in terms of $\partial G/\partial x$ and $\partial G/\partial y$. On the other hand, this formula for $y'(x)$ in terms of $\partial G/\partial x$ and $\partial G/\partial y$ leads to the natural criterion for an affirmative answer to the existence question.

Example 15.6 Let's look at a specific example first. Consider the cubic implicit function

$$x^2 - 3xy + y^3 - 7 = 0 \tag{8}$$

around the point $x = 4,\ y = 3$. (Check that this point satisfies (8).) Suppose that we could find a function $y = y(x)$ which solves (8). Plugging this function into (8) yields

$$x^2 - 3xy(x) + y(x)^3 - 7 = 0.$$

Differentiate this expression with respect to x, using the product rule to differentiate the second term and the Chain Rule to differentiate the third term:

$$2x - 3y(x) - 3xy'(x) + 3y(x)^2 \cdot y'(x) = 0,$$

or

$$y'(x) = -\frac{2x - 3y}{3y^2 - 3x}. \tag{9}$$

At $x = 4, y = 3$, we find

$$y'(4) = -\frac{2 \cdot 4 - 3 \cdot 3}{3 \cdot 3^2 - 3 \cdot 4} = \frac{1}{15}.$$

We conclude that if there is a function $y(x)$ which solves (8) and if it is differentiable, then as x changes by Δx, the corresponding y will change by $\Delta x/15$.

Now, let's carry this computation out more generally for the implicit function $G(x, y) = c$ around the specific point $x = x_0, y = y_0$. We suppose that there is a

C^1 solution $y = y(x)$ to the equation $G(x, y) = c$, that is, that

$$G(x, y(x)) = c. \tag{10}$$

We will use the Chain Rule (Theorem 14.1) to differentiate (10) with respect to x at x_0:

$$\frac{\partial G}{\partial x}(x_0, y(x_0)) \cdot \frac{dx}{dx} + \frac{\partial G}{\partial y}(x_0, y(x_0)) \cdot \frac{dy}{dx}(x_0) = 0,$$

or

$$\frac{\partial G}{\partial x}(x_0, y_0) + \frac{\partial G}{\partial y}(x_0, y_0) \cdot y'(x_0) = 0.$$

Solving for $y'(x_0)$ yields

$$y'(x_0) = -\frac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)}. \tag{11}$$

We see from (11) that if the solution $y(x)$ of $G(x, y) = c$ exists and is differentiable. it is necessary that $(\partial G/\partial y)(x_0, y_0)$ be nonzero. As the following fundamental result of mathematical analysis indicates, this necessary condition is also a sufficient condition.

Theorem 15.1 **(Implicit Function Theorem)** Let $G(x, y)$ be a C^1 function on a ball about (x_0, y_0) in $\mathbf{R}^2$. Suppose that $G(x_0, y_0) = c$ and consider the expression

$$G(x, y) = c.$$

If $(\partial G/\partial y)(x_0, y_0) \neq 0$, then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that:

(*a*) $G(x, y(x)) \equiv c$ for all x in I,
(*b*) $y(x_0) = y_0$, and
(*c*) $y'(x_0) = -\dfrac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)}.$

Example 15.7 Consider the equation

$$G(x, y) \equiv x^2 - 3xy + y^3 - 7 = 0 \tag{12}$$

about the point $(x_0, y_0) = (4, 3)$ in Example 15.6. One computes that

$$\frac{\partial G}{\partial x} = 2x - 3y = -1 \qquad \text{at } (4, 3)$$

$$\frac{\partial G}{\partial y} = -3x + 3y^2 = 15 \qquad \text{at } (4, 3).$$

Since $(\partial G/\partial y)(4, 3) = 15 \neq 0$, Theorem 15.1 tells us that (12) does indeed define y as a C^1 function of x around $x_0 = 4$, $y_0 = 3$. Furthermore,

$$y'(x_0) = -\frac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)} = \frac{1}{15},$$

just as we discovered in Example 15.6. We can now conclude that the solution corresponding to $x_1 = 4.3$ is roughly

$$y_1 \approx y_0 + y'(x_0)\Delta x = 3 + \left(\frac{1}{15}\right) \cdot .3 = 3.02,$$

which compares well with the actual $y_1 = 3.01475\ldots$, which had to be computed numerically.

Example 15.8 Return to the equation $x^2 + y^2 = 1$. We saw that this equation does determine y as a function of x around the point $x = 0$ and $y = 1$. We can easily compute that $\partial G/\partial y = 2y = 2 \neq 0$ at $(0, 1)$. So, Theorem 15.1 assures us that $y(x)$ exists. Furthermore, it tells us that

$$y'(x)\Big|_{x=0} = -\frac{\partial G/\partial x}{\partial G/\partial y} = -\frac{2x}{2y} = -\frac{0}{2} = 0,$$

when $x = 0$ and $y = 1$. In this case, we have an explicit formula

$$y(x) = \sqrt{1 - x^2} \tag{13}$$

for $y(x)$. We can compute directly from (13) that

$$y'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

which does indeed equal zero when $x = 0$. Of course, we can see in Figure 15.1 that the graph of $y(x)$ is horizontal at (0, 1), so its derivative should be zero.

On the other hand,we noted in Figure 15.2 that no nice function $y(x)$ exists for $x^2 + y^2 = 1$ around $x = 1$, $y = 0$. This is consistent with Theorem 15.1, since $\partial G/\partial y = 2y = 0$ at (1, 0).

Several Exogenous Variables in an Implicit Function

Theorem 15.1 and the discussion around it carry over in a straightforward way to the situation where there are many exogenous variables, but still one equation and therefore one endogenous variable:

$$G(x_1, \dots, x_k, y) = c. \tag{14}$$

Around a given point $(x_1^*, \dots, x_k^*, y^*)$, we want to vary $\mathbf{x} = (x_1, \dots, x_k)$ and then find a y-value which corresponds to each such $(x_1, \dots, x_k)$. In this case, we say that equation (14) defines y as an **implicit function** of $(x_1, \dots, x_k)$. Once again, given G and $(\mathbf{x}^*, y^*)$, we want to know whether this functional relationship exists and, if it does, how does y change if any of the x_i's change from x_i^*. Since we are working with a function of several variables $(x_1, \dots, x_k)$, we will hold all but one of the x_i's constant and vary one exogenous variable at a time. But this puts us right back in the two-variable situation that we have been discussing.

The natural extension of Theorem 15.1 to this setting is the following.

Theorem 15.2 **(Implicit Function Theorem)** Let $G(x_1, \dots, x_k, y)$ be a C^1 function around the point $(x_1^*, \dots, x_k^*, y^*)$. Suppose further that $(x_1^*, \dots, x_k^*, y^*)$ satisfies

$$G(x_1^*, \dots, x_k^*, y^*) = c$$

and that

$$\frac{\partial G}{\partial y}(x_1^*, \dots, x_k^*, y^*) \neq 0.$$

Then, there is a C^1 function $y = y(x_1, \dots, x_n)$ defined on an open ball B about $(x_1^*, \dots, x_k^*)$ so that:

(*a*) $G(x_1, \dots, x_k, y(x_1, \dots, x_k)) = c$ for all $(x_1, \dots, x_k) \in B$,
(*b*) $y^* = y(x_1^*, \dots, x_k^*)$, and
(*c*) for each index i,

$$\frac{\partial y}{\partial x_i}(x_1^*, \dots, x_k^*) = -\frac{\dfrac{\partial G}{\partial x_i}(x_1^*, \dots, x_k^*, y^*)}{\dfrac{\partial G}{\partial y}(x_1^*, \dots, x_k^*, y^*)}. \tag{15}$$

EXERCISES

15.1 *a*) Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$.

b) Then, estimate the y value which corresponds to $x = 4.8$.

15.2 Suppose that we want to solve $G(x, y) = c$ for x as a function of y around some point (x_0, y_0). Write out a careful statement of the Implicit Function Theorem to handle this case.

15.3 For equation (8), estimate y when $x = 3.7$.

15.4 Can you solve (8) for y as a function of x when $x = 0$. If so, estimate the y's that correspond to $x = -.1$ and to $x = .15$ respectively.

15.5 Use the implicit form and the explicit form to compute $y'(x)$ for $(x, y) = (1, 1)$ in Example 15.2.

15.6 Consider the function $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3$.

a) If $x_1 = 6$ and $x_2 = 3$, find a y which satisfies $F(x_1, x_2, y) = 0$.

b) Does this equation define y as an implicit function of x_1 and x_2 near $x_1 = 6$, $x_2 = 3$?

c) If so, compute $(\partial y/\partial x_1)(6, 3)$ and $(\partial y/\partial x_2)(6, 3)$.

d) If x_1 increases to 6.2 and x_2 decreases to 2.9, estimate the corresponding change in y.

15.7 Consider the profit-maximizing firm in Example 15.5. If p increases by Δp and w increases by Δw, what will be the corresponding effect on the optimal input amount x?

15.8 Consider the equation $x^3 + 3y^2 + 4xz^2 - 3z^2y = 1$. Does this equation define z as a function of x and y:

a) In a neighborhood of $x = 1, y = 1$?

b) In a neighborhood of $x = 1, y = 0$?

c) In a neighborhood of $x = 0.5, y = 0$? If so, compute $\partial z/\partial x$ and $\partial z/\partial y$ at this point.

15.9 Consider $3x^2yz + xyz^2 = 30$ as defining x as an implicit function of y and z around the point $x = 1, y = 3, z = 2$.

a) If y increases to 3.2 and z remains at 2, use the Implicit Function Theorem to estimate the corresponding x.

b) Use the quadratic formula to solve $3x^2yz + xyz^2 = 30$ for x as an explicit function of y and z. Use approximation by differentials on this explicit formula to estimate x when y is 3.2 and $z = 2$.

c) Which way was easier?

15.2 LEVEL CURVES AND THEIR TANGENTS

Geometric Interpretation of the Implicit Function Theorem

In this section, we look at the Implicit Function Theorem from a more geometric point of view. In general, we would expect that the equation $G(x, y) = c$ of two variables defines a curve in the plane. For example, the equation $Ax + By = C$

defines a line in the plane and the equation $x^2 + y^2 = 1$ defines a circle in the plane. We can view the Implicit Function Theorem as telling us the following geometric information:

> When the set of points in the plane which satisfy the equation $G(x, y) = c$ can be considered as the graph of a function $y = f(x)$ of one variable, especially in the neighborhood of some fixed solution (x_0, y_0).

Example 15.9 Consider again the equation $x^2 + y^2 = 1$, which describes a circle of radius 1. Figure 15.1 indicates that we can think of the arc of the circle about the point (0, 1) as the graph of a function $y = f(x)(= \sqrt{1 - x^2})$. However, as Figure 15.2 indicates, the arc of the circle about (1, 0) cannot be considered as the graph of a function $y = f(x)$. Such an f would be double-valued for x to the left of $x = 1$ and empty-valued for x to the right of $x = 1$.

In addition to telling us whether the locus $G(x, y) = c$ can be described as the graph of a function $y = f(x)$, the Implicit Function Theorem also tells us the slope $f'(x)$ of the tangent line to the graph at (x, y). Consequently, it tells us the slope of the curve at $G(x, y) = c$. We summarize this geometric interpretation of the Implicit Function Theorem as follows.

Theorem 15.3 Let (x_0, y_0) be a point on the locus of points $G(x, y) = c$ in the plane, where G is a C^1 function of two variables. If $(\partial G/\partial y)(x_0, y_0) \neq 0$, then $G(x, y) = c$ defines a smooth curve around (x_0, y_0) which can be thought of as the graph of a C^1 function $y = f(x)$. Furthermore, the slope of this curve is:

$$-\frac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)}$$

If $(\partial G/\partial y)(x_0, y_0) = 0$, but $(\partial G/\partial x)(x_0, y_0) \neq 0$, then the Implicit Function Theorem tells us that the locus of points $G(x, y) = c$ is a smooth curve about (x_0, y_0), which we can consider as defining x as a function of y. It also tells us that the tangent line to the curve at (x_0, y_0) is parallel to the y-axis, i.e., vertical.

Definition A point (x_0, y_0) is called a **regular point** of the C^1 function $G(x, y)$ if

$$\frac{\partial G}{\partial x}(x_0, y_0) \neq 0 \quad \text{or} \quad \frac{\partial G}{\partial y}(x_0, y_0) \neq 0.$$

If every point (x, y) on the locus $G(x, y) = c$ is a regular point of G, then we call the level set $\{(x, y) : G(x, y) = c\}$ a **regular curve** or sometimes a one-dimensional **manifold**.

If $G(x, y) = c$ is a regular curve in the plane, then Theorem 15.3 states that at each point on the curve, the curve can be considered as defining y as a function of x or x as a function of y. Furthermore, there is a well-defined tangent line at each point on this curve.

Proof Sketch

The Implicit Function Theorem is so important that we would be remiss to avoid a discussion of its proof. So, we now sketch a proof of the above geometric version of the Implicit Function Theorem — Theorem 15.3.

Let G be a C^1 function on $\mathbf{R}^2$, as in the statement of Theorem 15.3. We suppose that $G(x_0, y_0) = 0$ and that $(\partial G/\partial y)(x_0, y_0) \neq 0$; without loss of generality, we assume that $(\partial G/\partial y)(x_0, y_0) > 0$. Since G is C^1, $(\partial G/\partial y)$ is continuous and we can find an $\varepsilon > 0$ and a small square

$$S \equiv \{(x, y) : x_0 - \varepsilon \leq x \leq x_0 + \varepsilon,\ y_0 - \varepsilon \leq y \leq y_0 + \varepsilon\}$$

for which $(\partial G/\partial y)(x, y) > 0$ for all $(x, y) \in S$. For $x_1 \in (x_0 - \varepsilon, x_0 + \varepsilon)$, let ℓ_{x_1} denote the vertical line segment in S through (x_1, y_0), as in Figure 15.3:

$$\ell_{x_1} \equiv \{(x, y) : x = x_1,\ y_0 - \varepsilon \leq y \leq y_0 + \varepsilon\} \subset S.$$

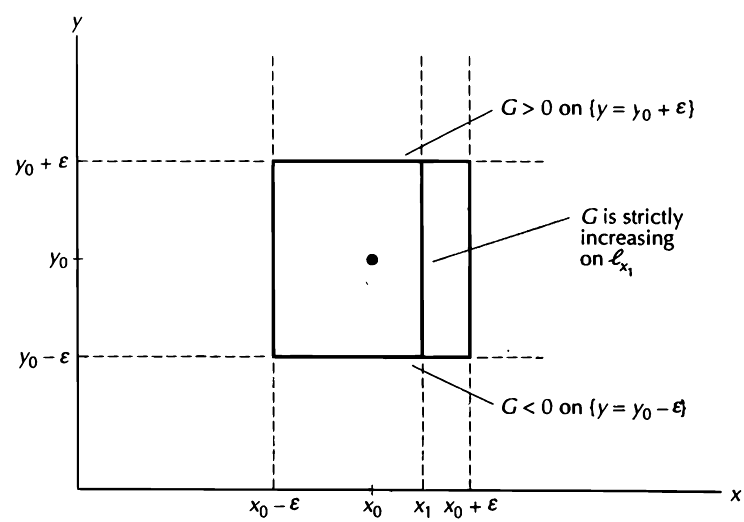


Figure 15.3 *The square S in $\mathbf{R}^2$.*

Since $(\partial G/\partial y)$ is positive on each ℓ_x, G is strictly increasing on each ℓ_x. Since $G(x_0, y_0) = 0$ and G is strictly increasing on ℓ_{x_0}, $G(x_0, y_0 - \varepsilon) < 0$ and $G(x_0, y_0 + \varepsilon) > 0$. By the continuity of $\partial G/\partial y$, we can choose the ε in the definition of S small enough so that

$$G(x, y_0 - \varepsilon) < 0 \quad \text{and} \quad G(x, y_0 + \varepsilon) > 0 \quad \text{for all } x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

In other words, G is negative on the bottom side of the square S and positive on the top side of S. On each vertical segment ℓ_x:

(1) G is negative at the bottommost point $(x, y_0 - \varepsilon)$,
(2) G is positive at the topmost point $(x, y_0 + \varepsilon)$, and
(3) G is strictly increasing.

It follows that for each x in $(x_0 - \varepsilon, x_0 + \varepsilon)$, there is a unique $y = y(x)$, depending on x, for which $G(x, y) = 0$. A little more work shows that the continuity of G implies that the dependency of $y(x)$ on x is continuous and that the differentiability of G implies that the dependency of $y(x)$ on x is differentiable. This $y(x)$ is the smooth function in the conclusion of Theorem 15.3.

Relationship to the Gradient

In Section 14.6, we learned that the gradient vector $\nabla G(x, y)$ of a C^1 function G points into the direction of greatest increase. Now, we will prove a complementary result: that the gradient is always perpendicular to the level curve; that is, it is perpendicular to the tangent line to the level curve at (x, y). Of course, in order to make this assertion, we need to guarantee that the level curve of G through (x_0, y_0) really does have a tangent line. By the Implicit Function Theorem, we need only require that (x_0, y_0) be a regular point of G.

Theorem 15.4 Let G be a C^1 function on a neighborhood of (x_0, y_0). Suppose that (x_0, y_0) is a regular point of G. Then, the gradient vector $\nabla G(x_0, y_0)$ is perpendicular to the level set of G at (x_0, y_0).

Proof Let (x_0, y_0) be a regular point of G:

$$\nabla G(x_0, y_0) = \left(\frac{\partial G}{\partial x}(x_0, y_0), \frac{\partial G}{\partial y}(x_0, y_0)\right) \neq (0, 0).$$

If $(\partial G/\partial y)(x_0, y_0) = 0$, then the gradient is a horizontal vector and the tangent to the level set is a vertical line, as we saw above. In this case, the two are perpendicular to each other. In general, the slope of the level set of G through

(x_0, y_0) is

$$-\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}.$$

The vector which realizes this slope is

$$\mathbf{v} = \left(1, -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}\right)$$

Since

$$\mathbf{v} \cdot \nabla G(x_0, y_0) = \left(1, -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}\right) \cdot \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right) = 0,$$

$\mathbf{v}$ and $\nabla G(x_0, y_0)$ point in perpendicular directions. ∎

Example 15.10 The gradient of $G(x, y) = x^2 + y^2$ is the vertical vector $(0, 2)$ at the point $(0, 1)$, where the circle is horizontal; and it is the horizontal vector $(2, 0)$ at the point $(1, 0)$ where the circle is vertical. See Figure 15.4.

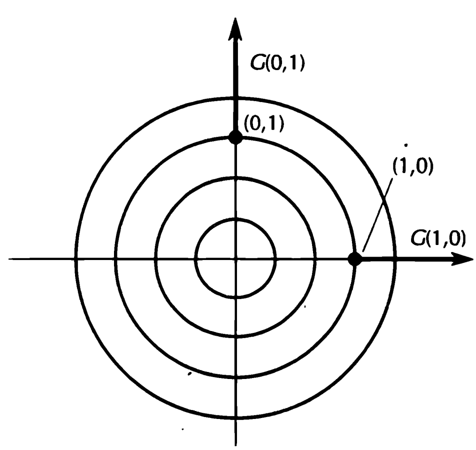


Figure 15.4 *Gradients of $G(x, y) = x^2 + y^2$.*

The geometry behind the statement of Theorem 15.4 gives a geometric justification of the hypotheses of the Implicit Function Theorem. If $G(x, y) = c$ defines a regular curve around the point (x_0, y_0), this curve will be the graph of a function $y = f(x)$ if and only if the curve is not vertical at (x_0, y_0), that is, if and only if the gradient is not horizontal at (x_0, y_0), that is, if and only if the y-component $\partial G/\partial y$, of $\nabla G(x_0, y_0)$ is not zero.

Tangent to the Level Set Using Differentials

We present one more piece of evidence for the conclusion of the Implicit Function Theorem that the slope of the level set $G(x, y) = c$ at (x_0, y_0) is $-(\partial G/\partial x)(x_0, y_0)/(\partial G/\partial y)(x_0, y_0)$. In Section 14.4, we used differentials to approximate the change in G in the vicinity of (x_0, y_0):

$$G(x_0 + \Delta x, y_0 + \Delta y) - G(x_0, y_0) \approx \frac{\partial G}{\partial x}(x_0, y_0)\Delta x + \frac{\partial G}{\partial y}(x_0, y_0)\Delta y. \tag{16}$$

We can use (16) to ask what combinations of linear movements Δx and Δy from (x_0, y_0) lead to *no change* in G. This should be the direction of the tangent line to the level set $\{G(x, y) = G(x_0, y_0)\}$ at (x_0, y_0). To find this direction, just set ΔG, the left hand side of (16), equal to zero:

$$0 = \frac{\partial G}{\partial x}(x_0, y_0)\Delta x + \frac{\partial G}{\partial y}(x_0, y_0)\Delta y. \tag{17}$$

The direction of no change in G at (x_0, y_0) is given by solving (17) for $\Delta y/\Delta x$:

$$\frac{\Delta y}{\Delta x} = -\frac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)} \tag{18}$$

We can use expressions (17) and (18) to restate Theorems 15.3 and 15.4 in terms of the tangent directions to the level set of G at (x_0, y_0).

Theorem 15.5 Let G be a C^1 function on a neighborhood of (x_0, y_0). Suppose that (x_0, y_0) is a regular point of G. Then, the vector $\mathbf{v} = (v_1, v_2)$ points in the direction parallel to the tangent line to the level set of G at (x_0, y_0) if and only if

$$DG(x_0, y_0)\,\mathbf{v} = \frac{\partial G}{\partial x}(x_0, y_0)\,v_1 + \frac{\partial G}{\partial y}(x_0, y_0)\,v_2 = 0$$

that is, $\mathbf{v}$ is in the nullspace of $DG(x_0, y_0)$.

This use of the Implicit Function Theorem is the natural approach when studying the slope of an indifference curve of a utility function and the slope of an isoquant of a production function, since in these situations we really are interested in which directions to move to keep the function constant. Recall that the level curve of a utility function $U(x, y)$ is called an **indifference curve** of U. Its slope at (x_0, y_0) is called the **marginal rate of substitution** (**MRS**) of U at (x_0, y_0) since it measures, in a marginal sense, how much more of good y the consumer would require to compensate for the loss of one unit of good x to keep the same level of satisfaction. By the Implicit Function Theorem, the MRS at (x_0, y_0) is:

$$-\frac{\dfrac{\partial U}{\partial x}(x_0, y_0)}{\dfrac{\partial U}{\partial y}(x_0, y_0)}$$

Similarly, if $Q = F(K, L)$ is a production function, its level curves are called **isoquants** and the slope $-F_K/F_L$ of an isoquant at (K_0, L_0) is called the **marginal rate of technical substitution** (**MRTS**). It measures how much of one input would be needed to compensate for a one-unit loss of the other unit while keeping production at the same level.

Level Sets of Functions of Several Variables

For a function $F(x_1, \dots, x_n)$ of more than two variables, the level sets will in general be $(n - 1)$-dimensional objects. For example, the level set $Ax + By + Cz = D$ is a two-dimensional plane in $\mathbf{R}^3$, and the level set $x^2 + y^2 + z^2 = 1$ is a two-dimensional sphere of radius 1, as pictured in Figure 15.5. On both of these sets, at each point there are two independent directions in which one can move. If some $(\partial F/\partial x_i)(\mathbf{x}^*) \neq 0$, then the Implicit Function Theorem tells us that the level set of F through $\mathbf{x}^*$ can be considered as the graph of a function of x_i in terms of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ around $\mathbf{x}^*$ in $\mathbf{R}^n$:

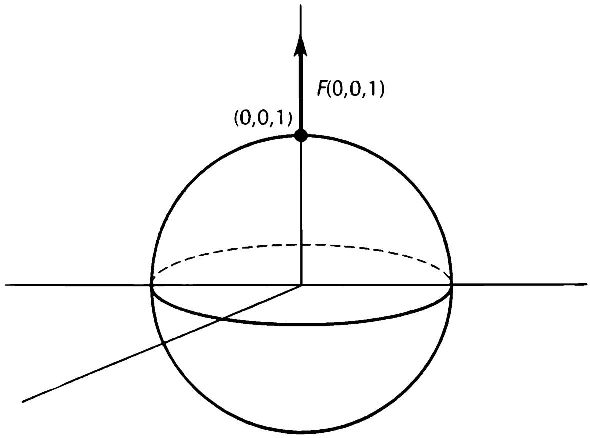
$$x_i = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

In this case, the tangent hyperplane to the level set of F is the tangent hyperplane to the graph of f. As in two dimensions, the gradient vector

$$\nabla F(\mathbf{x}^*) = \left(\frac{\partial F}{\partial x_1}(\mathbf{x}^*), \dots, \frac{\partial F}{\partial x_n}(\mathbf{x}^*)\right)$$

is perpendicular to the tangent hyperplane of the level set.

Example 15.11 The point (0, 0, 1) is the "north pole" on the sphere $x^2 + y^2 + z^2 = 1$. The gradient vector there is (0, 0, 2) which points due north, perpendicular to the sphere at (0, 0, 1), as illustrated in Figure 15.5.



The sphere $x^2 + y^2 + z^2 = 1$ *in* $\mathbf{R}^3$.

Figure 15.5

Example 15.12 By the discussion in Section 10.6, the vector

$$\mathbf{n} = (A, B, C)$$

is perpendicular (or normal) to the plane

$$Ax + By + Cy = D$$

at every point on the plane. Since $\mathbf{n}$ is also the gradient vector ∇F of $F(x, y, z) = Ax + By + Cz$, we see that ∇F is perpendicular to the level set $F(x, y, z) = D$, as we computed above.

For future reference, we summarize the analogue of Theorem 15.4 for implicit functions of several variables. First, we extend the definition of a regular curve to define a regular surface.

Definition A point $\mathbf{x}^*$ is called a **regular point** of the C^1 function $F(x_1, \ldots, x_n)$ if $\nabla F(\mathbf{x}^*) \neq 0$, that is, if some $(\partial F/\partial x_i)(\mathbf{x}^*)$ is not zero. If every point on the level set

$$\mathcal{F}_c \equiv \{(x_1, \ldots, x_n) : F(x_1, \ldots, x_n) = c\}$$

is a regular point of F, then we call $\mathcal{F}_c$ a **regular surface** or $(n-1)$-dimensional **manifold** in $\mathbf{R}^n$.

Theorem 15.6 If $F: \mathbf{R}^n \to \mathbf{R}^1$ is a C^1 function, if $\mathbf{x}^*$ is a point in $\mathbf{R}^n$, and if some $(\partial F/\partial x_i)(\mathbf{x}^*) \neq 0$, then:

(*a*) the level set of F through $\mathbf{x}^*$,

$$\mathcal{J}_{F(\mathbf{x}^*)} \equiv \{(x_1, \dots, x_n) : F(x_1, \dots, x_n) = F(\mathbf{x}^*)\}, \tag{19}$$

can be viewed as the graph of a real-valued C^1 function of $(n - 1)$ variables in a neighborhood of $\mathbf{x}^*$;

(*b*) the gradient vector $\nabla F(\mathbf{x}^*)$, considered as a vector at $\mathbf{x}^*$, is perpendicular to the tangent hyperplane of $\mathcal{J}_{F(\mathbf{x}^*)}$ at $\mathbf{x}^*$; and

(*c*) the vector $\mathbf{v}$, as a vector with its tail at $\mathbf{x}^*$, is a tangent vector to the level set (19) at $\mathbf{x}^*$ if and only if $\mathbf{v}$ is in the nullspace of $DF(\mathbf{x}^*)$; that is, $DF(\mathbf{x}^*)\mathbf{v} = \mathbf{0}$.

EXERCISES

15.10 *a*) For $(x, y) = (1, 1), (1, 0)$, and $(-2, 1)$, draw the level sets of $f(x, y) = x^2 + y^2$ through (x, y) and the gradient vector of f at (x, y).

b) Repeat this process for $f(x, y) = x^2 - y^2$.

15.11 *a*) Write an equation involving the partial derivatives of $f(x, y)$ and $g(x, y)$ that is equivalent to the condition that the level curves of f and g intersect only at right angles.

b) Show that the level curves intersect orthogonally if $f_x = g_y$ and $f_y = -g_x$.

15,12 Consider the function $f(x, y) = x^2 e^y$.

a) What is the slope of the level set at $x = 2, y = 0$?

b) In what direction should one move from the point (2, 0) in order to increase f most quickly? Express your answer as a vector of length 1.

15.13 A firm uses x hours of unskilled labor and y hours of skilled labor each day to produce $Q(x, y) = 60x^{2/3}y^{1/3}$ units of output per day. It currently employs 64 hours of unskilled labor and 27 hours of skilled labor.

a) What is its current output?

b) In what direction (expressed as a unit vector) should it change (x, y) if it wants to increase output most rapidly?

c) The firm is planning to hire an additional hour and a half of skilled labor. Use calculus to estimate the corresponding change in unskilled labor that would keep its output at its current level.

15.3 SYSTEMS OF IMPLICIT FUNCTIONS

Definition A set of m equations in $m + n$ unknowns

$$\begin{array}{ccc} G_1(x_1, \dots, x_{m+n}) & = & c_1 \\ \vdots \quad \vdots & & \vdots \\ G_m(x_1, \dots, x_{m+n}) & = & c_m \end{array} \tag{20}$$

is called a system of **implicit functions** if there is a partition of the variables into exogenous variables and endogenous variables, so that if one substitutes into (20) numerical values for the exogenous variables, the resulting system can be solved uniquely (in some sense) for corresponding values of the endogenous variables. This is the natural generalization of the single-equation implicit function that we considered in Section 15.1.

Linear Systems

The last section of Chapter 7 discussed *linear* implicit systems and concluded that, for such systems, in order for each choice of values of the exogenous variables to determine a unique set of values of the endogenous equations, it is necessary and sufficient that:

(1) the number of endogenous variables equal the number of equations, and
(2) the (square) matrix of coefficients corresponding to the endogenous variables be nonsingular.

Example 15.13 Consider the linear system of implicit functions

$$\begin{aligned} 4x + 2y + 2z - r + 3s &= 5 \\ 2x + 2z + 8r - 5s &= 7 \\ 2x + 2y + r - s &= 0. \end{aligned} \tag{21}$$

Since there are three equations, we need three endogenous variables and therefore two exogenous variables. Let's try to work with y, z and r as endogenous and x and s as exogenous. Putting the exogenous variables on the right side and the endogenous variables on the left, we rewrite (21) as

$$\begin{pmatrix} 2 & 2 & -1 \\ 0 & 2 & 8 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 5 - 4x - 3s \\ 7 - 2x + 5s \\ -2x + s \end{pmatrix} \tag{22}$$

Since the determinant of the coefficient matrix in (22) is 40, we can invert (22) and solve for (y, z, r) *explicitly* in terms of x and s:

$$\begin{pmatrix} y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 0 & 2 & 8 \\ 2 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 - 4x - 3s \\ 7 - 2x + 5s \\ -2x + s \end{pmatrix}$$

or

$$\begin{pmatrix} y \\ z \\ r \end{pmatrix} = \frac{1}{40} \begin{pmatrix} 2 & -2 & 18 \\ 16 & 4 & -16 \\ -4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 5 - 4x - 3s \\ 7 - 2x + 5s \\ -2x + s \end{pmatrix}$$

On the other hand, if we want x, y, and z to be endogenous, we have to solve the system

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 + r - 3s \\ 7 - 8r + 5s \\ 0 - r + s \end{pmatrix} \tag{23}$$

Since the determinant of the coefficient matrix in (23) is zero, we know that there are right-hand sides for which (23) cannot be solved for (x, y, z). For example, take $r = -5$ and $s = 0$. Then, (23) becomes

$$\begin{aligned} 4x + 2y + 2z &= 0 \\ 2x \qquad + 2z &= 47 \\ 2x + 2y \qquad &= 5. \end{aligned}$$

Adding the last two equations yields the inconsistent system:

$$\begin{aligned} 4x + 2y + 2z &= 0 \\ 4x + 2y + 2z &= 52. \end{aligned}$$

Since there is no solution in (x, y, z) for $(r, s) = (-5, 0)$, this partition into exogenous and endogenous variables does not work.

Example 15.14 A classical system of implicit functions in economics is the Keynesian linear IS-LM model:

$$\begin{aligned} Y &= C + I + G && \text{(GNP accounting identity)} \\ C &= a + b(Y - T) && \text{(consumption function)} \\ I &= i_0 - i_1 r && \text{(investment function)} \\ M^s &= c_1 Y - c_2 r && \text{(money market equilibrium),} \end{aligned}$$

where Y is GNP or national income, C is consumer consumption, I is investment, G is government spending, T is tax collection, M^s is money supply, r is the interest rate, and the other six lowercase letters stand for *positive* behavioral parameters, with $0 < b < 1$. We follow the standard method of substituting the second two equations into the first equation and simplifying to obtain the system

$$\begin{aligned} (1 - b)Y + i_1 r &= a + i_0 + G - bT \\ c_1 Y - c_2 r &= M^s \end{aligned} \tag{24}$$

The natural endogenous variables in this model are Y and r, the variables on the left-hand side of (24). The (Y, r) coefficient matrix in (24),

$$\begin{pmatrix} 1-b & i_1 \\ c_1 & -c_2 \end{pmatrix}, \tag{25}$$

has determinant $-c_2(1-b)-i_1c_1$, which is nonzero since $0 < b < 1$. Therefore, we can solve system (24) for Y and r; in this case, we can invert the matrix (25) to obtain the explicit solution

$$\begin{pmatrix} Y \\ r \end{pmatrix} = \frac{1}{c_2(1-b)+i_1c_1}\begin{pmatrix} c_2 & i_1 \\ c_1 & -(1-b) \end{pmatrix}\begin{pmatrix} a+i_0+G-bT \\ M^s \end{pmatrix}$$

Nonlinear Systems

The corresponding result for nonlinear systems follows from the usual calculus paradigm: linearize by taking the derivative, apply the linear theorem to this linearized system, and transfer these results back to the original nonlinear system. We write the basic nonlinear system of m equations in $m+n$ unknowns as

$$\begin{aligned} F_1(y_1,\ldots,y_m,x_1,\ldots,x_n) &= c_1 \\ F_2(y_1,\ldots,y_m,x_1,\ldots,x_n) &= c_2 \\ \vdots \qquad\qquad & \quad \vdots \\ F_m(y_1,\ldots,y_m,x_1,\ldots,x_n) &= c_m, \end{aligned} \tag{26}$$

where we want $y_1,\ldots,y_m$ to be endogenous and $x_1,\ldots,x_n$ to be exogenous. From the linear theory, we know that there should be as many endogenous variables as there are independent equations, in this case m. The linearization of system (26) about the point $(\mathbf{y}^*, \mathbf{x}^*)$ is

$$\begin{aligned} \frac{\partial F_1}{\partial y_1}dy_1 + \cdots + \frac{\partial F_1}{\partial y_m}dy_m + \frac{\partial F_1}{\partial x_1}dx_1 + \cdots + \frac{\partial F_1}{\partial x_n}dx_n &= 0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\quad & \vdots \\ \frac{\partial F_m}{\partial y_1}dy_1 + \cdots + \frac{\partial F_m}{\partial y_m}dy_m + \frac{\partial F_m}{\partial x_1}dx_1 + \cdots + \frac{\partial F_m}{\partial x_n}dx_n &= 0, \end{aligned} \tag{27}$$

where all the partial derivatives are evaluated at the point $(\mathbf{y}^*, \mathbf{x}^*)$. By the Linear Implicit Function Theorem, the linear system (27) can be solved for $dy_1,\ldots,dy_m$

in terms of $dx_1, \ldots, dx_n$ if and only if the coefficient matrix of the dy_i's,

$$\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)} \equiv \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix} \tag{28}$$

is nonsingular at $(\mathbf{y}^*, \mathbf{x}^*)$. Because this system is linear, when the coefficient matrix (28) is nonsingular, we can use the inverse of (28) to solve the system (27) for the dy_i's in terms of the dx_j's and cverything else

$$\begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix} = -\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^n \frac{\partial F_1}{\partial x_i} dx_i \\ \vdots \\ \sum_{i=1}^n \frac{\partial F_m}{\partial x_i} dx_i \end{pmatrix} \tag{29}$$

Since the linear approximation (27) of the original system (26) is a true implicit function of the dy_i's in terms of the dx_j's, the basic principle of calculus leads us to the conclusion that the nonlinear system (26) defines the y_i's as implicit functions of the x_j's, at least in a neighborhood of $(\mathbf{y}^*, \mathbf{x}^*)$.

Furthermore, one can actually use the linear solution (29) of the dy_i's in terms of the dx_j's to find the derivatives of the y_i's with respect to the x_j's at $(\mathbf{x}^*, \mathbf{y}^*)$. To compute $\partial y_k/\partial x_h$ for some fixed indices h and k, recall that this derivative estimates the effect on y_k of a one unit increase in x_h ($dx_h = 1$). So, we set all the dx_j's equal to zero in (27) or (29) except dx_h and then we solve (27) or (29) for the corresponding dy_i's. If we use (29), we find

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_h} \\ \vdots \\ \frac{\partial y_m}{\partial x_h} \end{pmatrix} = -\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_h} \\ \vdots \\ \frac{\partial F_m}{\partial x_h} \end{pmatrix} \tag{30}$$

Alternatively, we can apply Cramer's rule to (27) and compute

$$\frac{\partial y_k}{\partial x_h} = -\frac{\det\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial x_h} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial x_h} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_k} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_k} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}} \tag{31}$$

$$\equiv -\frac{\det \dfrac{\partial(F_1, \ldots, F_k, \ldots, F_m)}{\partial(y_1, \ldots, x_h, \ldots, y_m)}}{\det \dfrac{\partial(F_1, \ldots, F_k, \ldots, F_m)}{\partial(y_1, \ldots, y_k, \ldots, y_m)}}.$$

The following theorem — the most general form of the Implicit Function Theorem — summarizes these conclusions.

Theorem 15.7 Let $F_1, \ldots, F_m : \mathbf{R}^{m+n} \to \mathbf{R}^1$ be C^1 functions. Consider the system of equations

$$\begin{aligned} F_1(y_1, \ldots, y_m, x_1, \ldots, x_n) &= c_1 \\ \vdots \qquad & \quad \vdots \\ F_m(y_1, \ldots, y_m, x_1, \ldots, x_n) &= c_m \end{aligned} \tag{32}$$

as possibly defining $y_1, \ldots, y_m$ as implicit functions of $x_1, \ldots, x_n$. Suppose that $(\mathbf{y}^*, \mathbf{x}^*)$ is a solution of (32). If the determinant of the $m \times m$ matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix} \equiv \frac{\partial(F_1, \ldots, F_h, \ldots, F_m)}{\partial(y_1, \ldots, y_h \ldots, y_m)}$$

evaluated at $(\mathbf{y}^*, \mathbf{x}^*)$ is nonzero, then there exist C^1 functions

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ \vdots \quad & \quad \vdots \\ y_m &= f_m(x_1, \ldots, x_n) \end{aligned} \tag{33}$$

defined on a ball B about $\mathbf{x}^*$ such that

$$\begin{aligned} F_1(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}), x_1, \ldots, x_n) &= c_1 \\ \vdots \qquad\qquad & \quad \vdots \\ F_m(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}), x_1, \ldots, x_n) &= c_m \end{aligned}$$

for all $\mathbf{x} = (x_1, \ldots, x_n)$ in B and

$$\begin{aligned} y_1^* &= f_1(x_1^*, \ldots, x_n^*) \\ \vdots \quad & \quad \vdots \\ y_m^* &= f_m(x_1^*, \ldots, x_n^*). \end{aligned}$$

Furthermore, one can compute $(\partial f_k/\partial x_h)(\mathbf{y}^*, \mathbf{x}^*) = (\partial y_k/\partial x_h)(\mathbf{y}^*, \mathbf{x}^*)$ by setting $dx_h = 1$ and $dx_j = 0$ for $j \neq h$ in (27) and solving the resulting system for dy_k. This can be accomplished:

(*a*) by inverting the nonsingular matrix (28) to obtain the solution (30) or
(*b*) by applying Cramer's rule to (27) to obtain the solution (31).

Example 15.15 Consider the system of equations

$$\begin{aligned} F_1(x, y, a) &\equiv x^2 + axy + y^2 - 1 = 0 \\ F_2(x, y, a) &\equiv x^2 + y^2 \quad - a^2 + 3 = 0 \end{aligned} \tag{34}$$

around the point $x = 0, y = 1, a = 2$. If we change a a little to a' near $a = 2$, can we find (x', y') near $(0, 1)$ so that (x', y', a') satisfies these two equations? To answer this question, we need the Jacobian of (F_1, F_2) with respect to the endogenous variables x and y at the point $x = 0, y = 1, a = 2$:

$$\det \begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{pmatrix} (0, 1, 2) = \det \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} = 4 \neq 0.$$

So, we can solve system (34) for x and y as functions of a near $(0, 1, 2)$.
Furthermore, at $x = 0, y = 1, a = 2$,

$$\frac{dy}{da} = -\frac{\det \dfrac{\partial(F_1, F_2)}{\partial(x, a)}}{\det \dfrac{\partial(F_1, F_2)}{\partial(x, y)}} = -\frac{\det \begin{pmatrix} 2x + ay & xy \\ 2x & -2a \end{pmatrix}}{\det \begin{pmatrix} 2x + ay & ax + 2y \\ 2x & 2y \end{pmatrix}},$$

and, plugging in (0, 1, 2),

$$\frac{dy}{da}(2) = -\frac{\det\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}}{\det\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}} = \frac{8}{4} = 2 > 0.$$

Therefore, if a increases to 2.1, the corresponding y will increase to about 1.2.

Let's use the other method to compute the effect on x. Take differentials of the nonlinear system

$$\begin{aligned}(2x + ay)\,dx + (ax + 2y)\,dy + xy\,da &= 0 \\ 2x\,dx + 2y\,dy - 2a\,da &= 0.\end{aligned}$$

Plug in $x = 0, y = 1, a = 2$:

$$\begin{aligned}2\,dx + 2\,dy &= 0\,da \\ 0\,dx + 2\,dy &= 4\,da.\end{aligned}$$

Clearly, $dy = 2\,da$ (as we just computed above) and $dx = -dy = -2\,da$. So, if a increases to 2.1, the corresponding x will decrease roughly to $-.2$.

Example 15.16 A natural nonlinear generalization of the linear IS-LM model in Example 15.14 is the system

$$\begin{aligned}Y &= C + I + G \\ C &= C(Y - T) \\ I &= I(r). \\ M^s &= M(Y, r),\end{aligned}$$

where the nonlinear functions $x \longmapsto C(x)$, $r \longmapsto I(r)$, and $(Y, r) \longmapsto M(Y, r)$ satisfy

$$0 < C'(x) < 1, \quad I'(r) < 0, \quad \frac{\partial M}{\partial Y} > 0, \quad \text{and} \quad \frac{\partial M}{\partial r} < 0. \tag{35}$$

The analogue to system (24) is

$$\begin{aligned}Y - C(Y - T) - I(r) &= G \\ M(Y, r) &= M^s,\end{aligned} \tag{36}$$

which we want to define Y and r as implicit functions of G, M^s, and T. Suppose that the current (G, M^s, T) is (G^*, M^{s*}, T^*) and that the corresponding (Y, r)-

equilibrium is (Y^*, r^*). If we vary (G, M^s, T) a little, is there a corresponding equilibrium (Y, r) and how does it change? The linearization of system (36) is

$$(1 - C'(Y^* - T^*))\, dY - I'(r^*)\, dr = dG - C'(Y^* - T^*)\, dT$$

$$\frac{\partial M}{\partial Y} dY + \frac{\partial M}{\partial r} dr = dM^s$$

or

$$\begin{pmatrix} 1 - C'(Y^* - T^*) & -I'(r^*) \\ \dfrac{\partial M}{\partial Y} & \dfrac{\partial M}{\partial r} \end{pmatrix} \begin{pmatrix} dY \\ dr \end{pmatrix} = \begin{pmatrix} dG - C'(Y^* - T^*)\, dT \\ dM^s \end{pmatrix}, \tag{37}$$

all evaluated at $(Y^*, r^*, G^*, M^{s*}, T^*)$. The determinant of the coefficient matrix in (37),

$$D \equiv (1 - C'(Y^* - T^*))\frac{\partial M}{\partial r} + I'(r^*)\frac{\partial M}{\partial Y},$$

is negative by (35), and therefore is nonzero. By Theorem 15.7, the system (36) really does define Y and r as implicit functions of G, M^s, and T around $(Y^*, r^*, G^*, M^{s*}, T^*)$. Inverting (37), we compute

$$\begin{pmatrix} dY \\ dr \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \dfrac{\partial M}{\partial r} & I'(r) \\ -\dfrac{\partial M}{\partial Y} & 1 - C'(Y^* - T^*) \end{pmatrix} \begin{pmatrix} dG - C'(Y^* - T^*)dT \\ dM^s \end{pmatrix}.$$

If we increase government spending G, keeping M^s and T fixed, we find

$$dY = \frac{1}{D}\frac{\partial M}{\partial r}\, dG \quad \text{and} \quad dr = -\frac{1}{D}\frac{\partial M}{\partial Y}\, dG,$$

so that both Y and r increase.

EXERCISES

15.14 Carry out the calculations in Example 15.14.

15.15 For the linear and the nonlinear IS-LM models (24) and (36), how are the equilibrium Y and r affected by an increase in M^s? by an increase in T?

15.16 One solution of the system $x^3y - z = 1,\ x + y^2 + z^3 = 6$ is $x = 1, y = 2, z = 1$. Use calculus to estimate the corresponding x and y when $z = 1.1$.

15.17 Consider the system of equations

$$y^2 + 2u^2 + v^2 - xy = 15, \quad 2y^2 + u^2 + v^2 + xy = 38,$$

at the solution $x = 1, y = 4, u = 1, v = -1$. Think of u and v as exogenous and x and y as endogenous. Use calculus to estimate the values of x and y that correspond to $u = .9$ and $v = -1.1$.

15.18 One solution of the system

$$2x^2 + 3xyz - 4uv = 16, \qquad x + y + 3z + u - v = 10$$

is $x = 1, y = 2, z = 3, u = 0, v = 1$. If one varies u and v near their original values and plugs these new values into this system, can one find unique values of x, y and z that still satisfy this system? Explain.

15.19 Does the system $xz^3 + y^2v^4 = 2$, $xz + yvz^2 = 2$ define v and z as C^1 functions of x and y around the point (1, 1, 1, 1)? If so, find $\partial z/\partial x$, $\partial z/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ there.

15.20 Check that $x = 1, y = 4, u = 1, v = -1$ is a solution of the system

$$y^2 + 2u^2 + v^2 - xy = 15, \qquad 2y^2 + u^2 + v^2 + xy = 38.$$

If y increases to 4.02 and x stays fixed, does there exist a (u, v) near $(1, -1)$ which solves this system? If not, why not? If yes, estimate the new u and v.

15.21 The economy of Northern Saskatchewan is in equilibrium when the system of equations

$$2xz + xy + z - 2\sqrt{z} = 11 \qquad xyz = 6$$

is satisfied. One solution of this set of equations is $x = 3, y = 2, z = 1$, and Northern Saskatchewan is in equilibrium at this point. Suppose that the prime minister discovers that the variable z (output of beaver pelts) can be conrolled by simple decree.

a) If the prime minister raises z to 1.1, use calculus to estimate the change in x and y.

b) If x were in the control of the prime minister and not y or z, explain why you cannot use this method to estimate the effect of reducing x from 3 to 2.95.

15.22 Consider the system of equations

$$x + 2y + z = 5, \qquad 3x^2yz = 12,$$

as defining some endogenous variables in terms of some exogenous variables.

a) Divide the three variables into exogenous ones and endogenous ones in a neighborhood of $x = 2, y = 1, z = 1$ so that the Implicit Function Theorem applies.

b) If each of the exogenous variables in your answer to *a*) increases by 0.25, use calculus to estimate how each of the endogenous variables will change.

15.23 Consider the system of two equations in three unknowns: $x + 2y + z = 5$, $3x^2yz = 12$.

a) At the point $x = 2, y = 1, z = 1$, why can we treat z as an exogenous variable and x and y are the dependent variables?

b) If z rises to 1.2, use calculus to estimate the corresponding x and y.

15.24 A firm uses two inputs to produce its output via the Cobb-Douglas production function $z = x^a y^b$, where $a = b = .5$. Its current level of inputs is $x = 25$, $y = 100$. The firm will introduce a new technology that will change the b-exponent on its production function to $b = .504$, with no change to a. Use calculus to estimate the input combination which will keep the total output the same and the sum of the inputs the same. [Hint: Work with the system $x^a y^b = c$ (or better $a \ln x + b \ln y = \ln c$) and $x + y = 125$.]

15.25 Treat the linear IS-LM model at the beginning of Example 15.14 as four equations. What is the natural choice of endogenous variables? Can this four-equation system be solved for these endogenous variables in terms of the other variables?

15.4 APPLICATION: COMPARATIVE STATICS

Let's put the Implicit Function Theorem to work in the most basic microeconomic example of general equilibrium: a pure exchange economy with two consumers — numbered 1 and 2 — and two consumption goods — parameterized by x and y. We suppose that consumer 1 has initial endowment $(e_1, 0)$ and that consumer 2 has initial endowment $(0, e_2)$. To describe the consumers' preferences, let u_1 and u_2 be C^2, strictly concave ($u_i'' < 0$) functions of a single variable and let α be a scalar between 0 and 1. For $i = 1, 2$, we assume that consumer i's preferences over consumption bundles (x, y) are described by the utility function

$$U_i(x_i, y_i) \equiv \alpha u_i(x_i) + (1 - \alpha) u_i(y_i). \tag{38}$$

These U_i's include Cobb-Douglas utility functions. (Exercise.) Let p and q denote the price of a unit of good 1 and good 2, respectively. In this example, we will write the equations for the equilibrium prices and consumption bundles for this model and then study how these bundles are affected by changes in the consumers' initial endowments.

Consumer i wants to consume the bundle (x_i, y_i) that maximizes U_i subject to the affordability constraint

$$p x_i + q y_i = \text{value of initial endowment}. \tag{39}$$

As one learns in intermediate microeconomics, and as we will discuss more fully in Chapter 18, at the bundle of choice, the consumer's marginal rate of substitution between the two goods, that is, the consumer's *internal* relative valuation of the goods

$$\frac{\dfrac{\partial U_i}{\partial x_i}(x_i, y_i)}{\dfrac{\partial U_i}{\partial y_i}(x_i, y_i)} = \frac{\alpha u_i'(x_i)}{(1 - \alpha) u_i'(y_i)}, \tag{40}$$

must equal the price ratio p/q, the market's *external* relative valuation of the two goods. By (38), (39) and (40), the equations that describe the optimal choice for consumer 1 are

$$\frac{\alpha u_1'(x_1)}{(1-\alpha)u_1'(y_1)} = \frac{p}{q}, \tag{41}$$

$$px_1 + qy_1 = pe_1, \tag{42}$$

and the corresponding equations for consumer 2

$$\frac{\alpha u_2'(x_2)}{(1-\alpha)u_2'(y_2)} = \frac{p}{q}, \tag{43}$$

$$px_2 + qy_2 = qe_2. \tag{44}$$

Since we are dealing with a pure exchange economy, the total amounts of both commodities are fixed:

$$x_1 + x_2 = e_1, \tag{45}$$

$$y_1 + y_2 = e_2. \tag{46}$$

Equations (41) through (46) form a system of six equations in the six unknowns x_1, y_1, x_2, y_2, p, and q. As usual, all prices are relative: multiplying both prices by the same scalar does not change equations (41) through (46). To remove this ambiguity, we will set $q = 1$. In the language of economics, we are treating good 2 as the *numeraire*.

We can ignore equation (44) because it is implied by equations (42), (45) and (46). (Exercise.) The remaining five equations can be written as

$$\begin{aligned} \frac{\alpha}{1-\alpha}u_1'(x_1) - pu_1'(y_1) &= 0 \\ px_1 + y_1 - pe_1 &= 0 \\ \frac{\alpha}{1-\alpha}u_2'(x_2) - pu_2'(y_2) &= 0 \\ x_1 + x_2 - e_1 &= 0 \\ y_1 + y_2 - e_2 &= 0. \end{aligned} \tag{47}$$

We begin by setting both endowments equal to 1 : $e_1 = e_2 = 1$. In this case, the unique solution of system (47) is

$$\begin{aligned} x_1 &= y_1 = \alpha \\ x_2 &= y_2 = 1 - \alpha \\ p &= \frac{\alpha}{1-\alpha}. \end{aligned} \tag{48}$$

(Exercise.) We ask how a change in the initial endowment e_2 affects the equilibrium consumption bundles and prices (48), keeping e_1 fixed.

The linearization of system (48) is

$$\begin{aligned}
\frac{\alpha}{1-\alpha}u_1''(x_1)\,dx_1 - pu_1''(y_1)\,dy_1 - u_1'(y_1)\,dp &= 0 \\
p\,dx_1 + 1\,dy_1 + (x_1 - 1)\,dp &= 0 \\
\frac{\alpha}{1-\alpha}u_2''(x_2)\,dx_2 - p\,u_2''(y_2)\,dy_2 - u_2'(y_2)\,dp &= 0 \\
1\,dx_1 + 1\,dx_2 &= 0 \\
1\,dy_1 + 1\,dy_2 &= de_2.
\end{aligned} \tag{49}$$

The Implicit Function Theorem tells us that, if we can solve linear system (49) for dx_1, dx_2, dy_1, dy_2, dp, then we can compute $\partial x_1/\partial e_2$, $\partial x_2/\partial e_2$, $\partial y_1/\partial e_2$, $\partial y_2/\partial e_2$, and $\partial p/\partial e_2$.

The easiest way to solve system (49) is to solve the last two equations for dx_2 and dy_2:

$$dx_2 = -dx_1 \qquad dy_2 = de_2 - dy_1,$$

and substitute (48) and these expressions for dx_2 and dy_2 into the first three equations of (49):

$$\begin{aligned}
\frac{\alpha}{1-\alpha}u_1''(\alpha)\,dx_1 - \frac{\alpha}{1-\alpha}u_1''(\alpha)\,dy_1 - u_1'(\alpha)\,dp &= 0 \\
\frac{\alpha}{1-\alpha}\,dx_1 + dy_1 + (\alpha - 1)\,dp &= 0 \\
-\frac{\alpha}{1-\alpha}u_2''(1-\alpha)\,dx_1 + \frac{\alpha}{1-\alpha}u_2''(1-\alpha)\,dy_1 - u_2'(1-\alpha)\,dp & \\
= \frac{\alpha}{1-\alpha}u_2''(1-\alpha)\,de_2. &
\end{aligned}$$

Multiply the first equation through by $(1-\alpha)/u_1'(\alpha)$, the second equation through by $(1-\alpha)$, and the third equation through by $(1-\alpha)^2/\alpha u_2'(1-\alpha)$:

$$\begin{pmatrix}
\frac{\alpha u_1''(\alpha)}{u_1'(\alpha)} & -\frac{\alpha u_1''(\alpha)}{u_1'(\alpha)} & -(1-\alpha) \\
\alpha & 1-\alpha & -(1-\alpha)^2 \\
-\frac{(1-\alpha)u_2''(1-\alpha)}{u_2'(1-\alpha)} & \frac{(1-\alpha)u_2''(1-\alpha)}{u_2'(1-\alpha)} & -\frac{(1-\alpha)^2}{\alpha}
\end{pmatrix}
\begin{pmatrix} dx_1 \\ dy_1 \\ dp \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ \frac{(1-\alpha)u_2''(1-\alpha)}{u_2'(1-\alpha)}\,de_2 \end{pmatrix}. \tag{50}$$

Let
$$r_i(z) \equiv -\frac{z\,u_i''(z)}{u_i'(z)}. \tag{51}$$

Expression (51) is a measure of the concavity of u_i; in studies of portfolio choice, it is called the **Arrow-Pratt measure of relative risk aversion.** For our purposes, it suffices to know that $r_1(z)$ and $r_2(z)$ are strictly positive. Rewrite system (50) as

$$\begin{pmatrix} -r_1(\alpha) & r_1(\alpha) & -(1-\alpha) \\ \alpha & 1-\alpha & -(1-\alpha)^2 \\ r_2(1-\alpha) & -r_2(1-\alpha) & -\dfrac{(1-\alpha)^2}{\alpha} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dy_1 \\ dp \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -r_2(1-\alpha)de_2 \end{pmatrix}$$

This system can be solved using Cramer's rule to get

$$\begin{aligned} dx_1 &= \frac{-R_2(1-R_1)(1-\alpha)^2}{D}\,de_2 \\ dy_1 &= \frac{(1-\alpha)R_2[R_1(1-\alpha)+\alpha]}{D}\,de_2 \\ dp &= \frac{R_1R_2}{D}\,de_2, \end{aligned} \tag{52}$$

where
$$R_1 \equiv r_1(\alpha) > 0, \qquad R_2 \equiv r_2(1-\alpha) > 0$$

and
$$\begin{aligned} D &\equiv \frac{R_1(1-\alpha)^2}{\alpha} + R_2(1-\alpha) \\ &= -(1-\alpha)^2\left(\frac{u_1''(\alpha)}{u_1'(\alpha)} + \frac{u_2''(1-\alpha)}{u_2'(1-\alpha)}\right) > 0. \end{aligned}$$

By the Implicit Function Theorem,

$$\begin{aligned} \frac{\partial x_1}{\partial e_2} &= \frac{-R_2(1-R_1)(1-\alpha)^2}{D} \\ \frac{\partial x_2}{\partial e_2} &= \frac{R_2(1-R_1)(1-\alpha)^2}{D} \\ \frac{\partial y_1}{\partial e_2} &= \frac{(1-\alpha)R_2[R_1(1-\alpha)+\alpha]}{D} \\ \frac{\partial y_2}{\partial e_2} &= 1 - \frac{(1-\alpha)R_2[R_1(1-\alpha)+\alpha]}{D} \\ \frac{\partial p}{\partial e_2} &= \frac{R_1R_2}{D} \end{aligned} \tag{53}$$

We conclude that when the initial endowments are $e_1 = e_2 = 1$, an increase of e_2, the endowment of good 2, leads to a rise in the price of good 1 relative to good 2 ($\partial p/\partial e_2 > 0$), and a rise in the consumption of good 2 by consumer 1 ($\partial y_1/\partial e_2 > 0$). What happens to good 1 depends upon the utility functions.

EXERCISES

15.26 Show that the utility functions (38) include Cobb-Douglas preferences.
15.27 Show that equation (44) is implied by equations (42), (45) and (46).
15.28 Show that when $e_1 = e_2 = 1$, (48) is the unique solution of system (47).
15.29 Verify the expressions in (52) and (53).
15.30 Compute the exact partial derivatives in (53) for $u_1(z) = u_2(z) = \ln z$ and $\alpha = 1/2$.
15.31 Compute the comparative statics that results from a change in e_1, holding e_2 fixed.
15.32 Compute and interpret the comparative statics that results from an increase in α.

15.5 THE INVERSE FUNCTION THEOREM (optional)

In this section, we present one more approach to the Implicit Function Theorem. This approach presents another illustration of the basic paradigm of calculus, namely, that one can learn a lot about a nonlinear function from its linear approximation. In this context, to solve a problem about the behavior of a nonlinear function F in the vicinity of a given point $\mathbf{x}^*$, take the derivative $DF_{\mathbf{x}^*}$ of F at $\mathbf{x}^*$, use the tools of linear algebra to glean the appropriate information about the linear function $DF_{\mathbf{x}^*}$, and use the techniques of calculus to transfer this information back to the original F.

For example, suppose that F is a C^1 function from $\mathbf{R}^n$ to $\mathbf{R}^m$, that $\mathbf{b}_0$ is a given point in the target space $\mathbf{R}^m$, and that $\mathbf{x}_0$ is a solution of the system of equations

$$F(\mathbf{x}) = \mathbf{b}_0. \tag{54}$$

A basic question of equilibrium analysis is: what happens if we vary $\mathbf{b}_0$ a little to $\mathbf{b}_1$? Does the corresponding equation $F(\mathbf{x}) = \mathbf{b}_1$ still have a solution? If it does, how many solutions does it have?

The main purpose of Chapter 7 was to answer these questions for a *linear* system of equations

$$A\mathbf{x} = \mathbf{b}_0. \tag{55}$$

The answers depended on the size and rank of A.

> If A is $m \times n$, then (55) has a solution for every right-hand side $\mathbf{b}_0$ if and only if $m \leq n$ and the rank of A is m; (55) has *at most one* solution for every right hand side $\mathbf{b}_0$ if and only if $m \geq n$ and the rank of A is n.