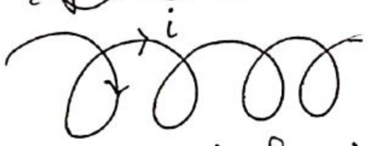
Calculation of Self-inductance

i) Long Solenoid :– (air cored)



Magnetic induction at a point inside long solenoid

$= B = \left(\frac{\mu_0 N I}{l}\right)$.

$N \rightarrow$ total number of turns
$l \rightarrow$ length of solenoid
$A \rightarrow$ area of cross-section.

Flux linked with each turn

$= \phi_1 = B \cdot A = \left(\frac{\mu_0 N I}{l}\right) A$

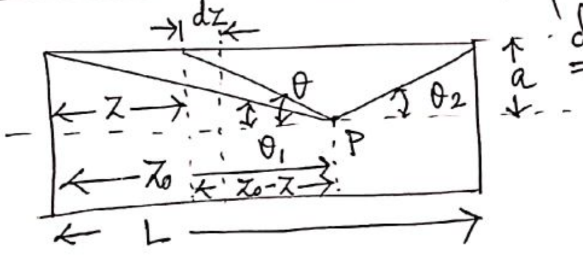
$\therefore$ flux linked with 'N' turns

$= \phi = N\phi_1 = \left(\frac{\mu_0 N^2 I}{l}\right) \cdot A$

$\therefore$ Self inductance $= L = \frac{\phi}{I} = \frac{\mu_0 N^2 A}{l}$ (Solenoid / Toroid)

Self inductance of solenoid / Toroidal coil of 'N' number of turns of mean length 'l'.

If solenoid is not very long :–



Magnetic induction at 'p' due to 'dz' this element is given by

$$dB = \left(\frac{N}{L}\right) dz \cdot \frac{\mu_0 I a^2}{2\left[a^2 + (z_0 - z)^2\right]^{3/2}}$$

$$= \left(\frac{N}{L}\right) \frac{\mu_0 I}{2} \times \frac{a}{\left[a^2 + (z_0 - z)^2\right]} \times \frac{a \quad dz}{\sqrt{a^2 + (z_0 - z)^2}}$$

$$= \left(\frac{N}{L}\right) \cdot \frac{\mu_0 I}{2} \cdot (\sin^2\theta) \cdot (\sin\theta) \cdot \frac{a\, d\theta}{(\sin^2\theta)}$$

$$= \frac{\mu_0 N I}{2L} \cdot \sin\theta\, d\theta.$$

$$\sin\theta = \frac{a}{\sqrt{a^2 + (z_0 - z)^2}}$$

$$\tan\theta = \frac{a}{(z_0 - z)}$$

or, $\frac{-a}{(z - z_0)} = \tan\theta$

or, $(z - z_0) = -a \cot\theta$

$$dz = (a\, \text{cosec}^2\theta)\, d\theta$$

$$= \frac{a}{\sin^2\theta}\, d\theta$$

$$\therefore B = \int_{\theta_1}^{\pi - \theta_2} \frac{\mu_0 N I}{2L} \sin\theta\, d\theta = \frac{\mu_0 N I}{2L} \cdot (\cos\theta_1 + \cos\theta_2)$$

$$= \frac{\mu_0 N I}{2L} \left[\frac{z_0}{\sqrt{z_0^2 + a^2}} + \frac{(L - z_0)}{\sqrt{a^2 + (L - z_0)^2}}\right]$$

For small length 'dz_0',

Number of turns is $\left(\frac{N}{L}\right) \cdot dz_0$

$\therefore$ Flux linking these turns

$$d\phi = \frac{B(\pi a^2)}{}\left(\frac{N}{L}\right) dz_0$$

variation of magnetic flux over cross-sectional area being neglected

$\therefore$ Total flux linking 'N' turns is

$$\Phi = \int_{z_0=0}^{l} \frac{B\cdot(\pi a^2) N}{l} \cdot dz_0$$

$$= \int_{z_0=0}^{l} \left(\frac{\mu_0 NI}{2L}\right)\left[\frac{z_0}{\sqrt{z_0^2+a^2}} + \frac{(l-z_0)}{\sqrt{a^2+(l-z_0)^2}}\right](\pi a^2) \times \frac{N}{L} \cdot dz_0$$

$$= (\pi a^2) \mu_0 \frac{N^2 I}{2L^2} \cdot \left[\sqrt{a^2+z_0^2} - \sqrt{a^2+(l-z_0)^2}\right]_{z_0=0}^{L}$$

$$= (\mu_0 \pi a^2) \frac{IN^2}{2L^2}\left[\sqrt{a^2+L^2} - a - a + \sqrt{a^2+L^2}\right]$$

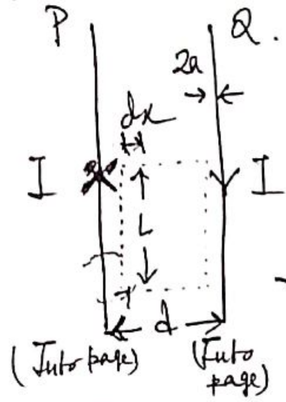
$$= \frac{\mu_0 N^2 (\pi a^2)}{l^2}\left[\sqrt{a^2+L^2} - a\right] I$$

(Self inductance) $L = \frac{\mu_0 N^2 \cdot (\pi a^2)}{L^2}\left[\sqrt{a^2+L^2} - a\right]$

if, $l \gg a$, then, $\sqrt{a^2+L^2} \approx L$

$$\therefore L = \frac{(\mu_0 N^2)(\pi a^2)}{L^{\cancel{2}}} \times \cancel{L} \approx \frac{\mu_0 N^2 A}{L}$$

Self inductance of two long parallel wires :-



Let 'P' and Q be two long parallel wires separated by distance 'd'

Two wires carry same current 'I' in opposite direction

These two wires construct an open-air transmission line.

Radius 'a' of each wire, $\underline{a << d}$.

$\therefore$ Flux inside wires themselves can be neglected

The flux is concentrated betⁿ region betⁿ two wires.

Consider rectangular area

length $\rightarrow L$, width dx, at 'x' distance from wire

$\therefore$ Flux through rectangle

$$d\varphi = B L\, dx$$

$$\therefore \varphi = \int_{x=a}^{d-a} \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{(d-x)}\right) l\,.dx$$

$$= \frac{\mu_0 I L}{2\pi}\cdot\Big[\ln(x) - \ln(d-x)\Big]_{x=a}^{d-a}$$

$$= \frac{\mu_0 I L}{2\pi}\left[\ln\left(\frac{x}{d-x}\right)\right]_{x=a}^{(d-a)}$$

$$= \frac{\mu_0 I L}{2\pi}\left[\ln\left(\frac{d-a}{a}\right) - \ln\left(\frac{a}{d-a}\right)\right]$$

$$= \frac{\mu_0 I L}{\pi}\ln\left(\frac{d-a}{a}\right) = L I$$

~~both wires~~ ~~lines points~~ flux lines adds up in antiparallel currents But it cancels in ~~opposite~~ parallel currents

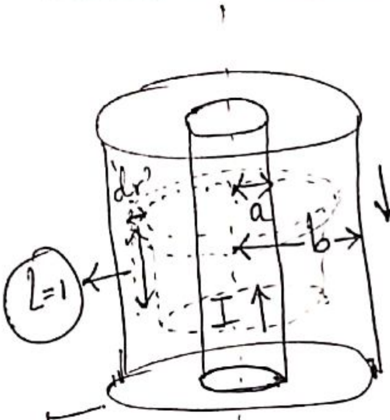
Since, $d >> a$, Self-inductance per unit length

$$\boxed{\left(L/l\right) = \frac{\mu_0}{\pi}\ln\left(d/a\right)}$$

if wires are very close together, $(d-a) \approx a$.

$\therefore L \approx 0$. This is the principle of $\underline{\text{NON-INDUCTIVE WINDING}}$, used to construct resistance coils. ~~Insulated~~ Insulated coil is doubled on itself before it is wound on $\underline{\text{format}}$.

Coaxial. Cylinders :-



Two Co-axial cylinders of radii 'a' and 'b' where (b > a), Carry same current 'I' in opposite direction.

They thus form Co-axial Cable.

Outside two cylinder mag. field is Zero.

If we consider a circle of radius 'r' $r > b$, for concentric Cable,

Total Current $I - I = 0$

if $a < r < b$, (and $b \gg a$)

$$B = \left(\frac{\mu_0 I}{2\pi r}\right)$$

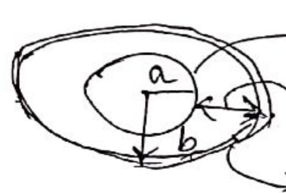
$$\text{Flux} = \int_{r=a}^{b} \left(\frac{\mu_0 I}{2\pi r}\right) (dr.1)$$ ← Imagine two Co-axial Cylinders of radii r and $r+dr$. of unit length ($l = 1$)

$$= \frac{\mu_0 I}{2\pi} \ln (b/a)$$

∴ 'L' (Inductance) per unit length $= \frac{\mu_0}{2\pi} \ln (b/a)$

Here, we have neglected flux inside materials of two Cylinders. Justified, when 1) ($b \gg a$)

2) Thickness of outer cylinder is very thin.



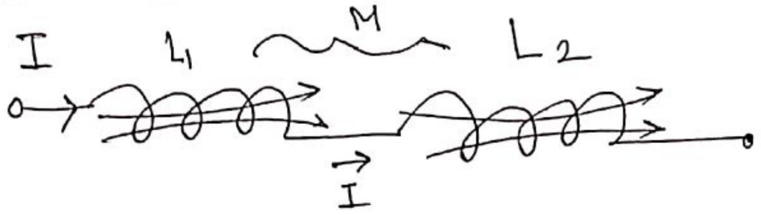
Core metal

(insulating layer)

Thin conducting layer

(Construction of Co-axial Cable)

Inductances in Series



E.m.f in coil 1

$$\mathcal{E}_{11} = -L_1\left(\frac{dI}{dt}\right)$$

Emf induced in coil 2 due to current I in coil 1

$$\mathcal{E}_{21} = -M\left(\frac{dI}{dt}\right)$$, $M \rightarrow$ Mutual inductance betn coils

$\therefore$ Em.f induced in coil 2

$$\mathcal{E}_{22} = -L_2\left(\frac{dI}{dt}\right)$$

Em.f induced in coil 1. due to current I in coil 2

$$\mathcal{E}_{12} = -M\left(\frac{dI}{dt}\right)$$

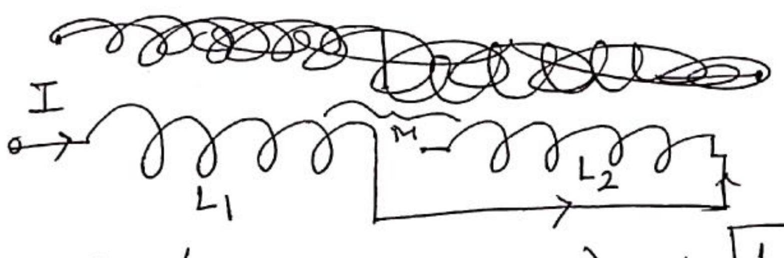
Net emf $= \mathcal{E} = \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{12} + \mathcal{E}_{21}$

$$= -(L_1 + L_2 + 2M)\left(\frac{dI}{dt}\right)$$

$$\therefore \quad \mathcal{E} = -L_{eq}\left(\frac{dI}{dt}\right).$$

$$\therefore \boxed{L_{eq} = L_1 + L_2 + 2M}$$

For anti-series connection



$$\mathcal{E} = (\mathcal{E}_{11} + \mathcal{E}_{12} - \mathcal{E}_{12} - \mathcal{E}_{21}) \quad \therefore \boxed{L_{eq} = L_1 + L_2 - 2M}$$

if $M = 0$, $\boxed{L_{eq} = L_1 + L_2}$ ✓

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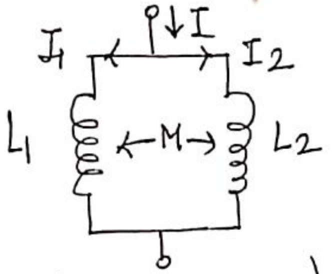
~~Parallel~~ Inductances Parallel



$$\mathcal{E}_1 = -L_1\left(\frac{dI_1}{dt}\right) - M\left(\frac{dI_2}{dt}\right)$$

$$\mathcal{E}_2 = -L_2\left(\frac{dI_2}{dt}\right) - M\left(\frac{dI_1}{dt}\right).$$

As coils are parallel $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$

$$\left.\begin{aligned} \therefore\ +L_1\left(\frac{dI_1}{dt}\right) + M\left(\frac{dI_2}{dt}\right) &= -\mathcal{E} \\ M\left(\frac{dI_1}{dt}\right) + L_2\left(\frac{dI_2}{dt}\right) &= -\mathcal{E} \end{aligned}\right\}$$

$$\therefore \left(\frac{dI_1}{dt}\right) = \frac{\begin{vmatrix} -\mathcal{E} & M \\ -\mathcal{E} & L_2 \end{vmatrix}}{\begin{vmatrix} L_1 & M \\ M & L_2 \end{vmatrix}} = \frac{-\mathcal{E}(L_2 - M)}{(L_1L_2 - M^2)}$$

$$\left(\frac{dI_2}{dt}\right) = \frac{\begin{vmatrix} L_1 & -\mathcal{E} \\ M & -\mathcal{E} \end{vmatrix}}{\begin{vmatrix} L_1 & M \\ M & L_2 \end{vmatrix}} = \frac{-\mathcal{E}(L_1 - M)}{(L_1L_2 - M^2)}$$

Now, $\left(\frac{dI}{dt}\right) = \left(\frac{dI_1}{dt}\right) + \left(\frac{dI_2}{dt}\right) = -\mathcal{E}\frac{(L_1 + L_2 - 2M)}{(L_1L_2 - M^2)}$

As, $\mathcal{E} = -L_{eq}\left(\frac{dI}{dt}\right) = -L_{eq} \times \left\{\frac{-\mathcal{E}(L_1 + L_2 - 2M)}{(L_1L_2 - M^2)}\right\}$

$$\therefore \boxed{L_{eq} = \frac{L_1L_2 - M^2}{(L_1 + L_2 - 2M)}} \checkmark$$

If Mutual inductance opposes self inductance

$$\boxed{L_{eq} = \frac{L_1L_2 - M^2}{(L_1 + L_2 + 2M)}} \checkmark$$

If $M = 0$, $\boxed{L_{eq} = \frac{L_1L_2}{L_1 + L_2}} \checkmark$

<u>Coefficient of coupling</u> :– Mutual flux betn two coils may be less, at ~~best~~ best equal to self fluxes of two coils.

$\therefore\ \phi_{12} \leq \phi_{22}$ and $\phi_{21} \leq \phi_{11}$

$\therefore\ \phi_{12} = K_1\phi_{22}$ and $\phi_{21} = K_2\phi_{11}$

$$\left.\begin{aligned} \therefore\ MI_2 = \phi_{12} = K_1\phi_{22} = K_1L_2I_2 &\Rightarrow M = K_1L_2 \\ \text{and}\ MI_1 = \phi_{21} = K_2\phi_{11} = K_2L_1I_1 &\Rightarrow M = K_2L_1 \end{aligned}\right\} \begin{aligned} &\therefore M^2 = K_1K_2L_1L_2 \\ &\therefore M = K\sqrt{L_1L_2} \end{aligned}$$

when, $0 \leq K \leq 1$, 'K' → "coefficient of coupling" of loops.

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20 unit

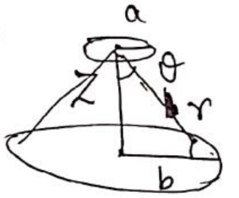
Q) A small loop of wire (radius a) lies a distance 'Z' above the Center of a large loop (radius b). The planes of two loops are parallel, and $\perp^r$ to the common axis.

a) Suppose Current 'I' flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop is essentially constant).

b) Suppose Current 'I' flows in the little loop. Find the flux through the big loop. (the little loop is so small that you may treat it as a magnetic dipole).

c) Find the mutual inductance and confirm $M_{12} = M_{21}$.

Proof:- a)



$\Rightarrow$ flux through little loop

$$\phi_{little} = \left[\frac{\mu_0 I b^2}{2(b^2+Z^2)^{3/2}}\right](\pi a^2)$$

$\hookrightarrow$ (Field along axis) of small loop.

b) Flux through big loop

$$\phi_{big} = \int B \cdot ds$$

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$ where m = magnetic moment $= (I\pi a^2)$ of little loop

$\hookrightarrow$ mag. field due to a ~~magnetic~~ small Current loop, which is equivalent to magnetic dipole.

$\leftarrow$ Integrating over "spherical Cap" bounded by big loop and centered at little loop

$$\therefore \phi_{big} = \frac{\mu_0 (I\cancel{\pi}a^2)}{\cancel{4\pi}_{2}\, r^3}\int_{\phi=0}^{2\pi}\int_0^{\theta}(2\cos\theta)(r^2\sin\theta)\,d\theta\,d\phi$$

$$= \frac{\mu_0 I a^2}{2r}\times 2\pi \times \frac{1}{\cancel{2}}\int_0^{\theta}\sin(2\theta)\,d\theta$$

$$= \frac{\mu_0 I a^2}{r}\times\frac{\pi}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\theta}$$

$$= \frac{\mu_0 I a^2}{r}\cdot\frac{\pi}{4}\times\left[1-\cos(2\theta)\right] = \frac{(\mu_0 I a^2)\pi}{4r}\left[2\sin^2\theta\right]$$

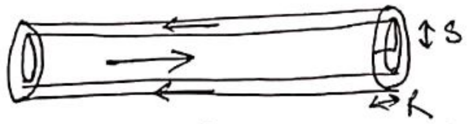
$$= \frac{(\mu_0 I a^2)\pi}{2}\times\frac{1}{r}\times\left(\frac{b}{r}\right)^2$$

$$= \frac{(\mu_0 I a^2)\pi}{2}\times\frac{b^2}{(Z^2+b^2)^{3/2}}$$

$$\therefore M_{12} = M_{21} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+Z^2)^{3/2}}$$

Q7.28 A long Cable Carries Current in one direction uniformly distributed over its (Circular) cross-section. The Current returns along the surface (there is very thin insulating sheath separating Currents). Find self inductance per unit length.

Solⁿ:-



(Radius of wire = R)

$$\oint \vec{B}\cdot\vec{dl} = B(2\pi s) = \mu_0 I_{enc} = \mu_0 I\left(\frac{s^2}{R^2}\right)$$

$$\therefore\ B = \mu_0 I\left(\frac{s^2}{R^2}\right)\times\left(\frac{1}{2\pi s}\right)$$

$$= \frac{\mu_0 I s}{2\pi R^2}$$

$$\therefore\ W = \frac{1}{2\mu_0}\int B^2\,dV$$

$$= \frac{1}{2\cancel{\mu_0}}\int_0^R \frac{\mu_0^{\cancel{2}} I^2 s^2}{4\pi^2 R^4}\cdot\underbrace{(2\pi s)\,l\,ds}_{\text{Cylindrical volume element}}$$

$$= \frac{\mu_0 I^2}{8\pi^{\cancel{2}} R^4}\times(\cancel{2}\cancel{\pi} l)\left[\frac{s^4}{\cancel{4}_2}\right]_0^R$$

$$= \frac{\mu_0 I^2}{16\pi}\times l\times\left(\frac{\cancel{R^4}}{\cancel{R^4}}\right) = \frac{1}{2}LI^2\ \text{(energy stored in coil)}$$

$$\therefore\ \boxed{L = \frac{\mu_0 l}{8\pi}} \Rightarrow \text{Self inductance per unit length} = \frac{L}{l} = \frac{\mu_0}{8\pi}\ \text{Sol}^n$$

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