Introduction to Maxwells eqns. Charge conservation Displacement current and resurraction by Eqn of Continuity

Electrodynamics Before Maxwell

i) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ (Gausses law)

ii) $\vec{\nabla}\cdot\vec{B} = 0$.

iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$ (Faradays law)

iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampere's law)

Inconsistency in these formulas.

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0 \quad \text{all fine}$$

But, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$

L.H.S $= 0$, But right hand side $\neq 0$ in general

For steady currents the divergence of $\vec{J}$ is zero. But beyond Magnetostatics. i.e varying current case Ampere's law cannot be right

In integral form Ampere's law reads

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

Eqn of Continuity

volume v bounded by surface S.

$$\frac{dq}{dt} = i = \oint_S \vec{J}\cdot d\vec{s} \quad \dots (1)$$

$\vec{J} \to$ volume current density.

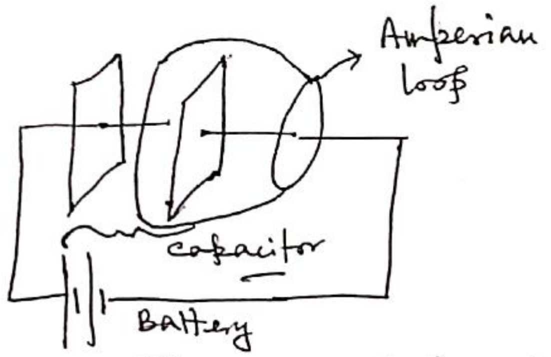
$$\therefore i = \oint \vec{J}\cdot d\vec{s} = -\frac{dq}{dt} = -\int\left(\frac{\partial\rho}{\partial t}\right)dv$$

Current is rate of decrease of charge per unit time as total charge is conserved.

$$\therefore \oint_S \vec{J}\cdot d\vec{s} + \int_v\left(\frac{\partial\rho}{\partial t}\right)dv = 0$$

$$\therefore \text{or, } \int_v(\vec{\nabla}\cdot\vec{J})\,dv + \int_v\left(\frac{\partial\rho}{\partial t}\right)dv = 0 \Rightarrow \boxed{\left(\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}\right) = 0}$$

$\hookrightarrow$ for arbitrary volume $dv \neq 0$.



We want to apply Ampere's law to shown loop.

How to find I_{enc}?

It is the total current passing through the loop, more precisely current piercing a surface that has the loop for its boundary. Here simplest surface lies in the plane of the loop — the wire punctures this surface, so $I_{enc} = I$.

But what if I draw the balloon shaped surface?

No current passes through this surface.

$\therefore \; I_{enc} = 0$

We never had this problem in magnetostatics as the conflict arises only when charge is piling up somewhere. (here the capacitor plates).

~~For nonsteady currents~~

Maxwells. Sol^n to problem

Applying eqn of continuity

$$\nabla \cdot J = -\left(\frac{\partial \rho}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\cdot\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Gauss's law

$\rho = \epsilon_0 \nabla \cdot \vec{E}$

$$\therefore \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

(So combining $\epsilon_0 \left(\frac{\partial E}{\partial t}\right)$ with J in Ampere's law would be right to kill the extra Divergence)

Such modification changes nothing as far magnetostatics is concerned.

When 'E' is constant, we still have $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$.

Maxwell's term is hard to detect in ordinary electromagnetic experiments, where it must compete for recognition with 'J'. Thus Faraday and others never discovered it in laboratory.

From $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$

The Maxwell's term demands, like Faradays law:- Change in magnetic field induces electric field.

$\Rightarrow$ (Changing Electric field induces a magnetic field)

Maxwell called his extra term Displacement Current

$$J_d \equiv \epsilon_0 \left(\frac{\partial E}{\partial t}\right).$$

Now to resolve the paradox of Capacitor charging.

If the Capacitor plates are very close together

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}. \quad \Bigg| \begin{array}{l} Q \rightarrow \text{charge on plate} \\ A \rightarrow \text{area}. \end{array}$$

$$\therefore \left(\frac{\partial E}{\partial t}\right) = \frac{1}{\epsilon_0 A}\left(\frac{dQ}{dt}\right) = \frac{1}{\epsilon_0 A}\cdot I$$

$$\therefore \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left(\frac{\partial E}{\partial t}\right) \cdot da$$

If we choose a flat Surface then $E = 0$ and $I_{enc} = I \longrightarrow$ (Comes from genuine current)

If we choose a Balloon-shaped Surface, then, $I_{enc} = 0$, but $\underbrace{\int \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a} = I/\epsilon_0}_{\text{(Comes from Displacement Current)}}$

$$\int_S \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a}$$

$$= \frac{d}{dt}\int_S (\vec{E}\cdot d\vec{a})$$

$$= \frac{d}{dt}\int_V (\vec{\nabla}\cdot\vec{E})\, dv$$

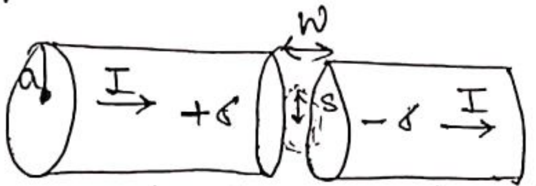
$$= \frac{d}{dt}\int_V \frac{\rho}{\epsilon_0}\, dv$$

$$= \frac{1}{\epsilon_0}\cdot\left(\frac{dq}{dt}\right) = \frac{I}{\epsilon_0}.$$

Displacement current related problem

2) A fat wire of radius 'a', carries a constant current 'I', uniformly distributed over its cross-section. A narrow gap in the wire of width $w \ll a$ forms a parallel plate ~~capacitor~~ capacitor, Find the magnetic field in the gap at a distance $s < a$ from the axis.

Solⁿ:-



Displacement current density

$$= J_d = \epsilon_0\left(\frac{\partial E}{\partial t}\right) = \frac{I}{A} = \frac{I}{\pi a^2}\hat{z}$$

Drawing an "amperian loop" at radius 's'

$$\oint \vec{B}\cdot d\vec{l} = B\cdot 2\pi s = \mu_0 I_{d\,enc}$$

$$= \mu_0\left(\frac{I}{\pi a^2}\right)(\pi s^2)$$

$$= \mu_0 I s^2/a^2$$

$$\therefore B = \frac{1}{2\pi s}\times\frac{\mu_0 I s^2}{a^2} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}.$$

2) 2) A thin wire that connect to the centers of the plates (forming capacitor plates)

Again current 'I' is constant, radius of capacitor is 'a', separation of the plates $w \ll a$.

Assume the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and zero at $t=0$.

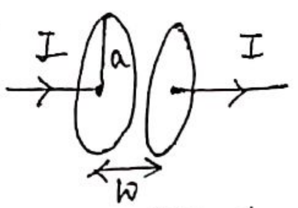
i) Find electric field betⁿ the plates as a function of 't'

ii) Find the displacement current through a circle of radius 's' in the plane midway betⁿ the plates. Using this circle as your "Amperian loop", and the flat surface that spans it, find the magnetic field at a distance 's' from the axis.

iii) Find displacement current through a cylindrical surface, which extends to the left through the plate and terminates outside the capacitor.

Displacement Current related problem

Q2) Solⁿ



$w << a$

Surface Current is uniform at 't'

Surface current zo at "t = 0"

i) $E = \frac{\sigma(t)}{\epsilon_0} \hat{z}$, $\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{I \cdot (t)}{\pi a^2}$

∴ Electric field Betⁿ plates $= \vec{E} = \frac{\sigma(t)}{\epsilon_0} \hat{z} = \frac{It}{\epsilon_0 \pi a^2} (\hat{z})$

ii) Displacement current through circle of radius 's' $= I_{denc} = I_d \cdot \pi s^2$

$$= \epsilon_0 \left(\frac{\partial E}{\partial t}\right) \cdot \pi s^2 = \epsilon_0 \frac{\partial}{\partial t}\left[\frac{It}{\epsilon_0 \pi a^2}\right] \cdot \pi s^2$$

$$= \cancel{\epsilon_0} \cdot \frac{I}{\cancel{\epsilon_0} \cancel{\pi} a^2} (\cancel{\pi} s^2)$$

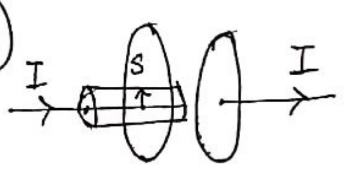
$$= I(s^2/a^2)$$

$$\therefore \oint \vec{B} \cdot \vec{dl} = \mu_0 I_{denc}$$

or, $(2\pi s) B = \mu_0 I (s^2/a^2)$

$\therefore B = \frac{\mu_0 I}{2\pi s} \times (s^2/a^2) = \left(\frac{\mu_0 I}{2\pi a^2}\right) s \hat{\phi}$. | Magnetic field at a distance 's' from the axis

iii)



A surface current flows radially outword over the left plate.

I(s) be total current crossing a circle of radius 's'.

∴ charge density at time (t) is

$$= \sigma(t) = \frac{[I - I(s)]\,t}{\pi s^2}$$

Since ~~this is~~ we are told this is $\sigma(t)$ is independent of 's',

$\therefore I - I(s) = \beta s^2$ (β is some constant)

But $I(a) = 0$, So, $\beta a^2 = I$ ∴ ~~$\beta = I$~~

$$\therefore \beta = (I/a^2) .$$

$$\therefore I(s) = I - \beta s^2 = I - \frac{I}{a^2} s^2 = I(1 - s^2/a^2)$$

$$\therefore B(2\pi s) = \mu_0 I_{enc} = \mu_0 [I - I(s)]$$

$$= (I - I + I s^2/a^2) \cdot \mu_0$$

$$\therefore B = \mu_0 I \frac{s^2}{a^2} \times \frac{1}{(2\pi s)} = \frac{\mu_0 I}{2\pi a^2} \cdot s \hat{\phi}$$

Displacement current related problem

Q3 If $E(s, t) = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) . \ln(a/s) \hat{z}$.

a) Find displacement current J_D

b) Integrate it to get total displacement current

$$I_d = \int \vec{J}_d . d\vec{a}$$

c) Compare I_d and I, (what is the ratio?)

If outer cylinder were 2mm in diameter, how high would be the frequency have to be for 'I_d' to be 1% of I?

Solⁿ :- a) $J_D = \epsilon_0 \left(\frac{\partial E}{\partial t}\right)$

$$= \epsilon_0 \frac{\mu_0 I_0 \omega^2}{2\pi} . \cos(\omega t) . \ln(a/s) \hat{z}$$

b) $$I_d = \int_0^a \frac{\epsilon_0 \mu_0 I_0 \omega^2}{\cancel{2\pi}} . \cos(\omega t) \ln(a/s) . (\cancel{2\pi} s) . ds$$

$$= \frac{\mu_0 \epsilon_0 \omega^2 I_0 \cos(\omega t)}{\cancel{2\pi}} \int_0^a [s \ln(a) - s \ln(s)] ds$$

$$= \mu_0 \epsilon_0 \omega^2 I_0 \cos(\omega t) \left\{ [\ln(a)] \frac{a^2}{2} - \left[\frac{s^2}{2} \ln(s) \Big|_0^a - \int_0^a \frac{1}{s} \frac{s^2}{2} . ds \right] \right\}$$

$$= \mu_0 \epsilon_0 \omega^2 I_0 \cos(\omega t) \left\{ \frac{a^2}{2} \cancel{\ln(a)} - \frac{a^2}{2} \cancel{\ln(a)} + \frac{a^2}{4} \right\}$$ ✓

c) $I_d / I = (\mu_0 \epsilon_0 \omega^2) \frac{a^2}{4}$, where $I = I_0 \cos(\omega t)$

$$= \frac{1}{c^2} \times \frac{\omega^2 a^2}{4} = \left(\frac{\omega a}{2c}\right)^2.$$

$a = 1\text{mm} = 10^{-3}$ m.

Now, $I_d / I = \frac{1}{100} = \left(\frac{\omega \times 10^{-3}}{2c}\right)^2$

$$\Rightarrow \frac{1}{10} = 10^{-1} = \frac{\omega \times 10^{-3}}{2c}$$

$$\therefore \omega = \frac{10^{-1} \times 2c}{10^{-3}} = 10^2 \times 2 \times 3 \times 10^8 = 6 \times 10^{10} \text{ Hz}$$

$$\sim 10^4 \text{ MHz}$$

∴ 'ω' has to be in microwave region

i.e. far above radiowave region.