6. Prove that $\underset{n\to\infty}{Lt}\left[\frac{1^m+2^m+\cdots+n^m}{n^{m+1}}\right]=\frac{1}{m+1}$

Ans.

$$\lim_{n\to\infty}\left[\frac{1^m+2^m+\cdots+n^m}{n^{m+1}}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^m+\left(\frac{2}{n}\right)^m+\cdots+\left(\frac{n}{n}\right)^m\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^m$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}(rh)^m \qquad \left[\text{taking } \frac{1}{n}=h\right]$$

[as $n\to\infty$, $h\to 0$]

$$=\int_0^1 x^m\,dx$$

$$=\left[\frac{x^{m+1}}{m+1}\right]_0^1=\frac{1}{m+1} \quad \text{(Ans.)}$$

5. Find the value of $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Ans.

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\therefore I + I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} dx = \pi/2$$

$\therefore 2I = \pi/2 \Rightarrow \boxed{I = \pi/4}$

3. Given $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$ find $\frac{d^2y}{dx^2}$

Ans. $x = t + \frac{1}{t}$

Differentiating both side we get

$$\frac{dx}{dt} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$$

$$y = t - \frac{1}{t}$$

Differentiating both side with respect to t, we get

$$\frac{dy}{dt} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$$

Now $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2+1}{t^2-1} = \frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= \frac{y - x\frac{dy}{dx}}{y^2}$$

$$= \frac{y - x \cdot \frac{x}{y}}{y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}}$$

4. Find the value of $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

Ans.

Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\, dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$\therefore I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} dx = \pi/2$$

$$2I = \pi/2 \Rightarrow \boxed{I = \pi/4}$$

1. Find $\frac{d^2y}{dx^2}$ when $(x+y)^{m+n} = x^m \cdot y^n$.

Ans. Given that
$$(x+y)^{m+n} = x^m \cdot y^n$$
taking log both sides, we have
$$(m+n)\log(x+y) = m\log x + n\log y$$
Differentiating with respect to x
$$\text{or } (m+n)\cdot\frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right) = m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx}$$
$$\text{or } \frac{dy}{dx}\left(\frac{m+n}{x+y} - \frac{n}{y}\right) = \frac{m}{x} - \frac{m+n}{x+y}$$
$$\text{or } \frac{dy}{dx}\cdot\frac{my+\cancel{ny}-nx-\cancel{ny}}{(x+y)\cdot y} = \frac{\cancel{mx}+my-\cancel{mx}-nx}{x\cdot(x+y)}$$
$$\text{or } \frac{dy}{dx}\cdot\frac{(my-nx)}{y} = \frac{(my-nx)}{x}$$
$$\text{or } \frac{dy}{dx} = \frac{y}{x} \quad \text{——(1)}$$
Differentiating with respect to x
$$\text{or } \frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y\cdot 1}{x^2}$$
$$\text{or } \frac{d^2y}{dx^2} = \frac{x\cdot y/x - y}{x^2} \quad [\text{by } (1)]$$
$$\text{or } \boxed{\frac{d^2y}{dx^2} = 0}$$

2. Prove that $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{\sqrt{n^2-1^2}}{n^2} + \cdots + \frac{\sqrt{n^2-(n-1)^2}}{n^2}\right] = \pi/4$

Ans.
$$\lim_{n\to\infty}\left[\frac{1}{n} + \frac{\sqrt{n^2-1^2}}{n^2} + \cdots + \frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$
$$= \lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2} + \frac{\sqrt{n^2-1^2}}{n^2} + \cdots + \frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$
$$= \lim_{n\to\infty}\frac{1}{n}\left[\sqrt{1-\left(\tfrac{0}{n}\right)^2} + \sqrt{1-\left(\tfrac{1}{n}\right)^2} + \cdots + \sqrt{1-\left(\tfrac{(n-1)}{n}\right)^2}\right]$$
$$= \lim_{\frac{1}{n}\to 0}\frac{1}{n}\left[\sqrt{1-\left(\tfrac{0}{n}\right)^2} + \sqrt{1-\left(\tfrac{1}{n}\right)^2} + \cdots + \sqrt{1-\left(\tfrac{(n-1)}{n}\right)^2}\right]$$
$$= \lim_{h\to 0} h\left[\sqrt{1-(0h)^2} + \sqrt{1-(1h)^2} + \cdots + \sqrt{1-((n-1)h)^2}\right]$$
$$= \lim_{h\to 0} h\sum_{r=0}^{n-1}\sqrt{1-(rh)^2}$$
$$= \int_0^1\sqrt{1-x^2}\,dx = \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$
$$= \pi/2\cdot\frac{1}{2} = \frac{\pi}{4} \ (\text{Ans}).$$