14.6 THE QUALITATIVE-GRAPHIC APPROACH

The several cases of nonlinear differential equations previously discussed (exact differential equations, separable-variable equations, and Bernoulli equations) have all been solved *quantitatively*. That is, we have in every case sought and found a time path $y(t)$ which, for each value of t, tells the specific corresponding value of the variable y.

At times, we may not be able to find a quantitative solution from a given differential equation. Yet, in such cases, it may nonetheless be possible to ascertain the *qualitative* properties of the time path—primarily, whether $y(t)$ converges—by directly observing the differential equation itself or by analyzing its graph. Even when quantitative solutions are available, moreover, we may still employ the techniques of qualitative analysis if the qualitative aspect of the time path is our principal or exclusive concern.

The Phase Diagram

Given a first-order differential equation in the general form

$$\frac{dy}{dt} = f(y)$$

either linear or nonlinear in the variable y, we can plot dy/dt against y as in Fig. 14.3. Such a geometric representation, feasible whenever dy/dt is a function of y alone, is called a *phase diagram*, and the graph representing the function f, a

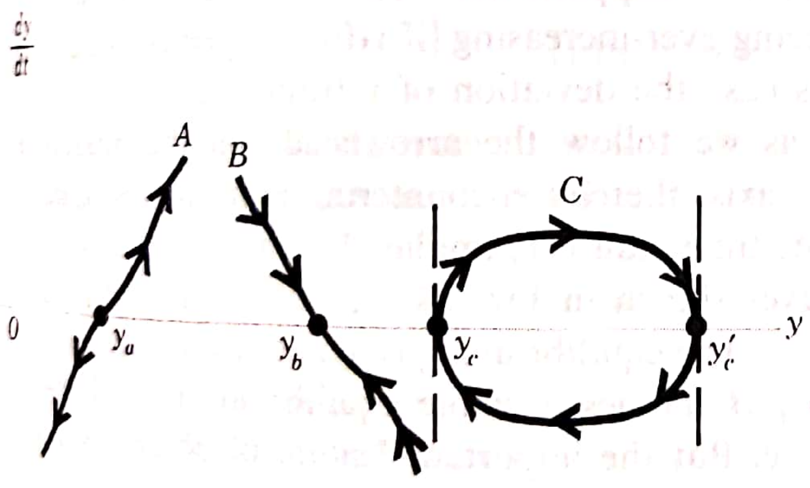

Figure 14.3

phase line. (A differential equation of this form—in which the time variable t does not appear as a separate argument of the function f—is said to be an *autonomous* differential equation.) Once a phase line is known, its configuration will impart significant qualitative information regarding the time path $y(t)$. The clue to this lies in the following two general remarks:

1. Anywhere *above* the horizontal axis (where $dy/dt > 0$), y must be increasing over time and, as far as the y axis is concerned, must be moving from left to right. By analogous reasoning, any point *below* the horizontal axis must be associated with a leftward movement in the variable y, because the negativity of dy/dt means that y decreases over time. These directional tendencies explain why the arrowheads on the illustrative phase lines in Fig. 14.3 are drawn as they are. Above the horizontal axis, the arrows are uniformly pointed toward the right—toward the northeast or southeast or due east, as the case may be. The opposite is true below the y axis. Moreover, these results are independent of the algebraic sign of y; even if phase line A (or any other) is transplanted to the left of the vertical axis, the direction of the arrows will not be affected.
2. An equilibrium level of y—in the intertemporal sense of the term—if it exists, can occur only on the horizontal axis, where $dy/dt = 0$ (y stationary over time). To find an equilibrium, therefore, it is necessary only to consider the intersection of the phase line with the y axis.* To test the dynamic stability of equilibrium, on the other hand, we should also check whether, regardless of the initial position of y, the phase line will always guide it toward the equilibrium position at the said intersection.

Types of Time Path

On the basis of the above general remarks, we may observe three different types of time path from the illustrative phase lines in Fig. 14.3.

Phase line A has an equilibrium at point y_a; but *above* as well as below that point, the arrowheads consistently lead away from equilibrium. Thus, although equilibrium can be attained if it happens that $y(0) = y_a$, the more usual case of $y(0) \neq y_a$ will result in y being ever-increasing [if $y(0) > y_a$] or ever-decreasing [if $y(0) < y_a$]. Besides, in this case the deviation of y from y_a tends to grow at an increasing pace because, as we follow the arrowheads on the phase line, we deviate farther from the y axis, thereby encountering ever-increasing numerical values of dy/dt as well. The time path $y(t)$ implied by phase line A can therefore be represented by the curves shown in Fig. 14.4a, where y is plotted against t (rather than dy/dt against y). The equilibrium y_a is dynamically unstable.

In contrast, phase line B implies a stable equilibrium at y_b. If $y(0) = y_b$, equilibrium prevails at once. But the important feature of phase line B is that

* However, not all intersections represent equilibrium positions. We shall see this when we discuss phase line C in Fig. 14.3.

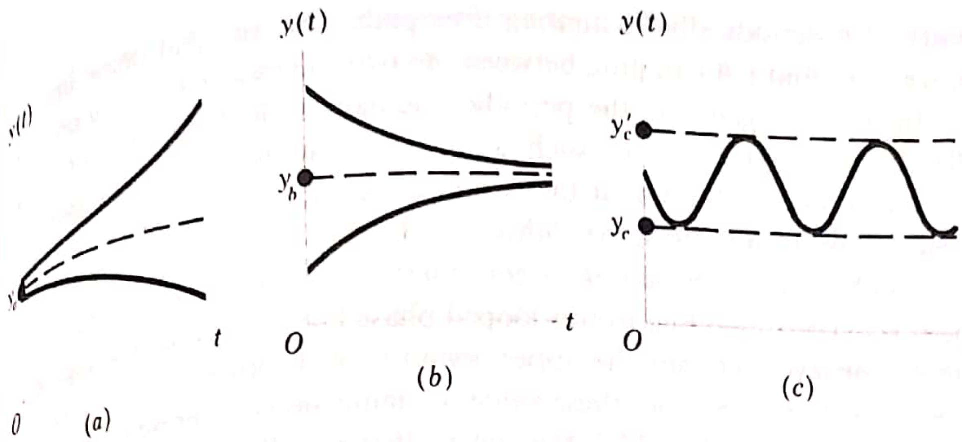

Figure 14.4

even if $y(0) \neq y_b$, the movement along the phase line will guide y toward the level of y_b. The time path $y(t)$ corresponding to this type of phase line should therefore be of the form shown in Fig. 14.4*b*, which is reminiscent of the dynamic market model.

The above discussion suggests that, in general, it is the slope of the phase line at its intersection point which holds the key to the dynamic stability of equilibrium or the convergence of the time path. A (finite) *positive* slope, such as at point y_a, makes for dynamic *instability*; whereas a (finite) *negative* slope, such as at y_b, implies dynamic *stability*.

This generalization can help us to draw qualitative inferences about given differential equations without even plotting their phase lines. Take the linear differential equation in (14.4), for instance:

$$\frac{dy}{dt} + ay = b \qquad \text{or} \qquad \frac{dy}{dt} = -ay + b$$

Since the phase line will obviously have the (constant) slope $-a$, here assumed nonzero, we may immediately infer (without drawing the line) that

$$a \gtrless 0 \quad \Leftrightarrow \quad y(t) \begin{Bmatrix} \text{converges to} \\ \text{diverges from} \end{Bmatrix} \text{equilibrium}$$

As we may expect, this result coincides perfectly with what the quantitative solution of this equation tells us:

$$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a} \qquad \text{[from (14.5')]}$$

We have learned that, starting from a nonequilibrium position, the convergence of $y(t)$ hinges on the prospect that $e^{-at} \to 0$ as $t \to \infty$. This can happen if and only if $a > 0$; if $a < 0$, then $e^{-at} \to \infty$ as $t \to \infty$, and $y(t)$ cannot converge. Thus, our conclusion is one and the same, whether it is arrived at quantitatively or qualitatively.

It remains to discuss phase line C, which, being a closed loop sitting across the horizontal axis, does not qualify as a *function* but shows instead a *relation* between dy/dt and y.* The interesting new element that emerges in this case is the

* This can arise from a second-degree differential equation $(dy/dt)^2 = f(y)$.

possibility of a periodically fluctuating time path. The way that phase line C is drawn, we shall find y fluctuating between the two values y_c and y'_c in a perpetual motion. In order to generate the periodic fluctuation, the loop must, of course, straddle the horizontal axis in such a manner that dy/dt can alternately be positive and negative. Besides, at the two intersection points y_c and y'_c, the phase line should have an infinite slope; otherwise the intersection will resemble either y_a or y_b, neither of which permits a continual flow of arrowheads. The type of time path $y(t)$ corresponding to this looped phase line is illustrated in Fig. 14.4c. Note that, whenever $y(t)$ hits the upper bound y'_c or the lower bound y_c, we have $dy/dt = 0$ (local extrema); but these values certainly do not represent equilibrium values of y. In terms of Fig. 14.3, this means that not all intersections between a phase line and the y axis are equilibrium positions.

In sum, for the study of the dynamic stability of equilibrium (or the convergence of the time path), one has the alternative either of finding the time path itself or else of simply drawing the inference from its phase line. We shall illustrate the application of the latter approach with the Solow growth model.

EXERCISE 14.6

1 Plot the phase line for each of the following, and discuss its qualitative implications:

(a) $\dfrac{dy}{dt} = y - 7$ (c) $\dfrac{dy}{dt} = 4 - \dfrac{y}{2}$

(b) $\dfrac{dy}{dt} = 1 - 5y$ (d) $\dfrac{dy}{dt} = 9y - 11$

2 Plot the phase line for each of the following and interpret:

(a) $\dfrac{dy}{dt} = (y+1)^2 - 16 \qquad (y \geq 0)$

(b) $\dfrac{dy}{dt} = \dfrac{1}{2}y - y^2 \qquad (y \geq 0)$

3 Given $dy/dt = (y-3)(y-5) = y^2 - 8y + 15$:

(a) Deduce that there are two possible equilibrium levels of y, one at $y = 3$ and the other at $y = 5$.

(b) Find the sign of $\dfrac{d}{dy}\left(\dfrac{dy}{dt}\right)$ at $y = 3$ and $y = 5$, respectively. What can you infer from these?

14.7 SOLOW GROWTH MODEL

The growth model of Professor Solow* is purported to show, among other things, that the razor's-edge growth path of the Domar model is primarily a result of the

* Robert M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, February, 1956, pp. 65–94.

SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS AND CONSTANT TERM

...gogic reasons, let us discuss first the method of solution for the ...der case ($n = 2$). The relevant differential equation is then the simple

$$y''(t) + a_1 y'(t) + a_2 y = b$$

..., a_2, and b are all constants. If the term b is identically zero, we have a homogeneous equation, but if b is a nonzero constant, the equation is *nonhomogeneous*. ...r discussion will proceed on the assumption that (15.2) is nonhomoge... ...r solving the nonhomogeneous version of (15.2), the solution of the homogeneous version will emerge automatically as a by-product.

...this connection, we recall a proposition introduced in Sec. 14.1 which is ...applicable here: If y_c is the *complementary function*, i.e., the general ... (with arbitrary constants) of the reduced equation of (15.2) and if y_p is ...*particular integral*, i.e., any particular solution (with no arbitrary constants) of ...complete equation (15.2), then $y(t) = y_c + y_p$ will be the general solution ...complete equation. As was explained previously, the y_p component provides ... the equilibrium value of the variable y in the intertemporal sense of the ..., whereas the y_c component reveals, for each point of time, the deviation of ...the path $y(t)$ from the equilibrium.

...Particular Integral

...the case of constant coefficients and constant term, the particular integral is ...ively easy to find. Since the particular integral can be *any* solution of (15.2), ...any value of y that satisfies this nonhomogeneous equation, we should always ...the simplest possible type: namely, $y =$ a constant. It $y =$ a constant, it ...ows that

$$y'(t) = y''(t) = 0$$

so that (15.2) in effect becomes $a_2 y = b$, with the solution $y = b/a_2$. Thus, the desired particular integral is

$$y_p = \frac{b}{a_2} \qquad (a_2 \neq 0) \tag{15.3}$$

Since the process of finding the value of y_p involves the condition $y'(t) = 0$, the rationale for considering that value as an intertemporal equilibrium becomes self-evident.

Example 1 Find the particular integral of the equation

$$y''(t) + y'(t) - 2y = -10$$

The relevant coefficients here are $a_2 = -2$ and $b = -10$. Therefore, the particular integral is $y_p = -10/(-2) = 5$.

What if $a_2 = 0$—so that the expression b/a_2 is not defined? In such a situation, since the constant solution for y_p fails to work, we must try some *nonconstant* form of solution. Taking the simplest possibility, we may try $y = kt$. Since $a_2 = 0$, the differential equation is now

$$y''(t) + a_1 y'(t) = b$$

but if $y = kt$, which implies $y'(t) = k$ and $y''(t) = 0$, this equation reduces to $a_1 k = b$. This determines the value of k as b/a_1, thereby giving us the particular integral

$$y_p = \frac{b}{a_1}t \qquad (a_2 = 0;\ a_1 \neq 0) \tag{15.3'}$$

Inasmuch as y_p is in this case a nonconstant function of time, we shall regard it as a moving equilibrium.

Example 2 Find the y_p of the equation $y''(t) + y'(t) = -10$. Here, we have $a_2 = 0$, $a_1 = 1$, and $b = -10$. Thus, by (15.3′), we can write

$$y_p = -10t$$

If it happens that a_1 is also zero, then the solution form of $y = kt$ will also break down, because the expression bt/a_1 will now be undefined. We ought, then, to try a solution of the form $y = kt^2$. With $a_1 = a_2 = 0$, the differential equation now reduces to the extremely simple form

$$y''(t) = b$$

and if $y = kt^2$, which implies $y'(t) = 2kt$ and $y''(t) = 2k$, the differential equation can be written as $2k = b$. Thus, we find $k = b/2$, and the particular integral is

$$y_p = \frac{b}{2}t^2 \qquad (a_1 = a_2 = 0) \tag{15.3''}$$

The equilibrium represented by this particular integral is again a moving equilibrium.

Example 3 Find the y_p of the equation $y''(t) = -10$. Since the coefficients are $a_1 = a_2 = 0$ and $b = -10$, formula (15.3″) is applicable. The desired answer is $y_p = -5t^2$.

The Complementary Function

The complementary function of (15.2) is defined to be the general solution of its reduced (homogeneous) equation

(15.4) $$y''(t) + a_1 y'(t) + a_2 y = 0$$

This is why we stated that the solution of a homogeneous equation will always be a *by-product* in the process of solving a complete equation.

Even though we have never tackled such an equation before, our experience with the complementary function of the first-order differential equations can supply us with a useful hint. From the solutions (14.3), (14.3′), (14.5), and (14.5′), it is clear that exponential expressions of the form Ae^{rt} figure very prominently in the complementary functions of first-order differential equations with constant coefficients. Then why not try a solution of the form $y = Ae^{rt}$ in the second-order equation, too?

If we adopt the trial solution $y = Ae^{rt}$, we must also accept

$$y'(t) = rAe^{rt} \qquad \text{and} \qquad y''(t) = r^2Ae^{rt}$$

as the derivatives of y. On the basis of these expressions for y, $y'(t)$, and $y''(t)$, the differential equation (15.4) can be transformed into

(15.4′) $$Ae^{rt}(r^2 + a_1 r + a_2) = 0$$

As long as we choose those values of A and r that satisfy (15.4′), the trial solution $y = Ae^{rt}$ should work. This means that we must either let $A = 0$ or see to it that r satisfies the equation

(15.4″) $$r^2 + a_1 r + a_2 = 0$$

Since the value of the (arbitrary) constant A is to be definitized by use of the initial conditions of the problem, however, we cannot simply set $A = 0$ at will. Therefore, it is essential to look for values of r that satisfy (15.4″).

Equation (15.4″) is known as the *characteristic equation* (or *auxiliary equation*) of the homogeneous equation (15.4), or of the complete equation (15.2). Because it is a quadratic equation in r, it yields two roots (solutions), referred to in the present context as *characteristic roots*, as follows:*

(15.5) $$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

These two roots bear a simple but interesting relationship to each other, which

* Note that the quadratic equation (15.4″) is in the normalized form; the coefficient of the r^2 term is 1. In applying formula (15.5) to find the characteristic roots of a differential equation, we must first make sure that the characteristic equation is indeed in the normalized form.

can serve as a convenient means of checking our calculation: The *sum* of the two roots is always equal to $-a_1$, and their *product* is always equal to a_2. The proof of this statement is straightforward:

$$r_1 + r_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}$$

$$+ \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} = \frac{-2a_1}{2} = -a_1 \tag{15.6}$$

$$r_1 r_2 = \frac{(-a_1)^2 - (a_1^2 - 4a_2)}{4} = \frac{4a_2}{4} = a_2$$

The values of these two roots are the only values we may assign to r in the solution $y = Ae^{rt}$. But this means that, in effect, there are *two* solutions which will work, namely,

$$y_1 = A_1 e^{r_1 t} \qquad \text{and} \qquad y_2 = A_2 e^{r_2 t}$$

where A_1 and A_2 are two arbitrary constants, and r_1 and r_2 are the characteristic roots found from (15.5). Since we want only *one* general solution, however, there seems to be one too many. Two alternatives are now open to us: (1) pick either y_1 or y_2 at random, or (2) combine them in some fashion.

The first alternative, though simpler, is unacceptable. There is only one arbitrary constant in y_1 or y_2, but to qualify as a general solution of a *second-order* differential equation, the expression must contain *two* arbitrary constants. This requirement stems from the fact that, in proceeding from a function $y(t)$ to its second derivative $y''(t)$, we "lose" two constants during the two rounds of differentiation; therefore, to revert from a second-order differential equation to the primitive function $y(t)$, two constants should be reinstated. That leaves us only the alternative of combining y_1 and y_2, so as to include both constants A_1 and A_2. As it turns out, we can simply take their *sum*, $y_1 + y_2$, as the general solution of (15.4). Let us demonstrate that, if y_1 and y_2, respectively, satisfy (15.4), then the sum $(y_1 + y_2)$ will also do so. If y_1 and y_2 are indeed solutions of (15.4), then by substituting each of these into (15.4), we must find that the following two equations hold:

$$y_1''(t) + a_1 y_1'(t) + a_2 y_1 = 0$$

$$y_2''(t) + a_1 y_2'(t) + a_2 y_2 = 0$$

By adding these equations, however, we find that

$$\underbrace{[y_1''(t) + y_2''(t)]}_{=\frac{d^2}{dt^2}(y_1 + y_2)} + a_1 \underbrace{[y_1'(t) + y_2'(t)]}_{=\frac{d}{dt}(y_1 + y_2)} + a_2(y_1 + y_2) = 0$$

Thus, like y_1 or y_2, the sum $(y_1 + y_2)$ satisfies the equation (15.4) as well. Accordingly, the general solution of the homogeneous equation (15.4) or the complementary function of the complete equation (15.2) can, in general, be written as $y_c = y_1 + y_2$.

A more careful examination of the characteristic-root formula (15.5) indicates, however, that as far as the values of r_1 and r_2 are concerned, three possible cases can arise, some of which may necessitate a modification of our result $y_c = y_1 + y_2$.

Case 1 (distinct real roots) When $a_1^2 > 4a_2$, the square root in (15.5) is a real number, and the two roots r_1 and r_2 will take *distinct* real values, because the square root is added to $-a_1$ for r_1, but subtracted from $-a_1$ for r_2. In this case, we can indeed write

$$y_c = y_1 + y_2 = A_1e^{r_1t} + A_2e^{r_2t} \qquad (r_1 \neq r_2) \tag{15.7}$$

Because the two roots are distinct, the two exponential expressions must be linearly independent (neither is a multiple of the other); consequently, A_1 and A_2 will always remain as separate entities and provide us with two constants, as required.

Example 4 Solve the differential equation

$$y''(t) + y'(t) - 2y = -10$$

The particular integral of this equation has already been found to be $y_p = 5$, in Example 1. Let us find the complementary function. Since the coefficients of the equation are $a_1 = 1$ and $a_2 = -2$, the characteristic roots are, by (15.5),

$$r_1, r_2 = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$$

(Check: $r_1 + r_2 = -1 = -a_1$; $r_1r_2 = -2 = a_2$.) Since the roots are distinct real numbers, the complementary function is $y_c = A_1e^t + A_2e^{-2t}$. Therefore, the general solution can be written as

$$y(t) = y_c + y_p = A_1e^t + A_2e^{-2t} + 5 \tag{15.8}$$

In order to definitize the constants A_1 and A_2, there is need now for *two* initial conditions. Let these conditions be $y(0) = 12$ and $y'(0) = -2$. That is, when $t = 0$, $y(t)$ and $y'(t)$ are, respectively, 12 and -2. Setting $t = 0$ in (15.8), we find that

$$y(0) = A_1 + A_2 + 5$$

Differentiating (15.8) with respect to t and then setting $t = 0$ in the derivative, we find that

$$y'(t) = A_1e^t - 2A_2e^{-2t} \qquad \text{and} \qquad y'(0) = A_1 - 2A_2$$

To satisfy the two initial conditions, therefore, we must set $y(0) = 12$ and $y'(0) = -2$, which results in the following pair of simultaneous equations:

$$A_1 + A_2 = 7$$
$$A_1 - 2A_2 = -2$$

with solutions $A_1 = 4$ and $A_2 = 3$. Thus the definite solution of the differential equation is

(15.8') $\quad y(t) = 4e^t + 3e^{-2t} + 5$

As before, we can check the validity of this solution by differentiation. The first and second derivatives of (15.8') are

$$y'(t) = 4e^t - 6e^{-2t} \qquad \text{and} \qquad y''(t) = 4e^t + 12e^{-2t}$$

When these are substituted into the given differential equation along with (15.8'), the result is an identity $-10 = -10$. Thus the solution is correct. As you can easily verify, (15.8') also satisfies both of the initial conditions.

Case 2 (repeated real roots) When the coefficients in the differential equation are such that $a_1^2 = 4a_2$, the square root in (15.5) will vanish, and the two characteristic roots take an identical value:

$$r(= r_1 = r_2) = -\frac{a_1}{2}$$

Such roots are known as *repeated roots*, or *multiple* (here, *double*) *roots*.

If we attempt to write the complementary function as $y_c = y_1 + y_2$, the sum will in this case collapse into a single expression

$$y_c = A_1e^{rt} + A_2e^{rt} = (A_1 + A_2)e^{rt} = A_3e^{rt}$$

leaving us with only one constant. This is not sufficient to lead us from a second-order differential equation back to its primitive function. The only way out is to find another eligible component term for the sum—a term which satisfies (15.4) and yet which is linearly independent of the term A_3e^{rt}, so as to preclude such "collapsing."

An expression that will satisfy these requirements is A_4te^{rt}. Since the variable t has entered into it multiplicatively, this component term is obviously linearly independent of the A_3e^{rt} term; thus it will enable us to introduce another constant, A_4. But does A_4te^{rt} qualify as a solution of (15.4)? If we try $y = A_4te^{rt}$, then, by the product rule, we can find its first and second derivatives to be

$$y'(t) = (rt + 1)A_4e^{rt} \qquad \text{and} \qquad y''(t) = (r^2t + 2r)A_4e^{rt}$$

Substituting these expressions of y, y', and y'' into the left side of (15.4), we get the expression

$$\left[(r^2t + 2r) + a_1(rt + 1) + a_2t\right]A_4e^{rt}$$

Inasmuch as, in the present context, we have $a_1^2 = 4a_2$ and $r = -a_1/2$, this last expression vanishes identically and thus is always equal to the right side of (15.4); this shows that A_4te^{rt} does indeed qualify as a solution.

Hence, the complementary function of the double-root case can be written as

(15.9) $\quad y_c = A_3e^{rt} + A_4te^{rt}$

Example 5 Solve the differential equation

$$y''(t) + 6y'(t) + 9y = 27$$

Here, the coefficients are $a_1 = 6$ and $a_2 = 9$; since $a_1^2 = 4a_2$, the roots will be repeated. According to formula (15.5), we have $r = -a_1/2 = -3$. Thus, in line with the result in (15.9), the complementary function may be written as

$$y_c = A_3e^{-3t} + A_4te^{-3t}$$

The general solution of the given differential equation is now also readily obtainable. Trying a constant solution for the particular integral, we get $y_p = 3$. It follows that the general solution of the complete equation is

$$y(t) = y_c + y_p = A_3e^{-3t} + A_4te^{-3t} + 3$$

The two arbitrary constants can again be definitized with two initial conditions. Suppose that the initial conditions are $y(0) = 5$ and $y'(0) = -5$. By setting $t = 0$ in the above general solution, we should find $y(0) = 5$; that is,

$$y(0) = A_3 + 3 = 5$$

This yields $A_3 = 2$. Next, by differentiating the general solution and then setting $t = 0$ and also $A_3 = 2$, we must have $y'(0) = -5$. That is,

$$y'(t) = -3A_3e^{-3t} - 3A_4te^{-3t} + A_4e^{-3t}$$

and

$$y'(0) = -6 + A_4 = -5$$

This yields $A_4 = 1$. Thus we can finally write the definite solution of the given equation as

$$y(t) = 2e^{-3t} + te^{-3t} + 3$$

Case 3 (complex roots) There remains a third possibility regarding the relative magnitude of the coefficients a_1 and a_2, namely, $a_1^2 < 4a_2$. When this eventuality occurs, formula (15.5) will involve the square root of a *negative* number, which cannot be handled before we are properly introduced to the concepts of *imaginary* and *complex* numbers. For the time being, therefore, we shall be content with the mere cataloging of this case and shall leave the full discussion of it to the next two sections.

The three cases cited above can be illustrated by the three curves in Fig. 15.1, each of which represents a different version of the quadratic function $f(r) = r^2 + a_1r + a_2$. As we learned earlier, when such a function is set equal to zero, the result is a quadratic *equation* $f(r) = 0$, and to solve the latter equation is merely to "find the zeros of the quadratic *function*." Graphically, this means that the roots of the equation are to be found on the horizontal axis, where $f(r) = 0$.

The position of the lowest curve in Fig. 15.1, is such that the curve intersects the horizontal axis twice; thus we can find two distinct roots r_1 and r_2, both of which satisfy the quadratic equation $f(r) = 0$ and both of which, of course, are

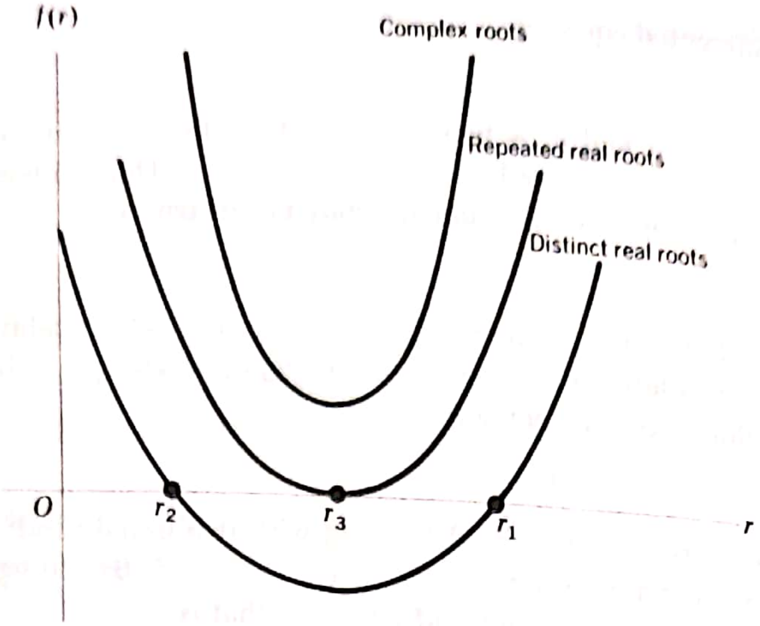

Figure 15.1

real-valued. Thus the lowest curve illustrates Case 1. Turning to the middle curve, we note that it meets the horizontal axis only once, at r_3. This latter is the only value of r that can satisfy the equation $f(r) = 0$. Therefore, the middle curve illustrates Case 2. Last, we note that the top curve does not meet the horizontal axis at all, and there is thus no real-valued root to the equation $f(r) = 0$. While there exist no real roots in such a case, there are nevertheless two complex numbers that can satisfy the equation, as will be shown in the next section.

The Dynamic Stability of Equilibrium

For Cases 1 and 2, the condition for dynamic stability of equilibrium again depends on the algebraic signs of the characteristic roots.

For Case 1, the complementary function (15.7) consists of the two exponential expressions $A_1 e^{r_1 t}$ and $A_2 e^{r_2 t}$. The coefficients A_1 and A_2 are arbitrary constants; their values hinge on the initial conditions of the problem. Thus we can be sure of a dynamically stable equilibrium ($y_c \to 0$ as $t \to \infty$), regardless of what the initial conditions happen to be, if and only if the roots r_1 and r_2 are *both* negative. We emphasize the word "both" here, because the condition for dynamic stability does *not* permit even *one* of the roots to be positive or zero. If $r_1 = 2$ and $r_2 = -5$, for instance, it might appear at first glance that the second root, being larger in absolute value, can outweigh the first. In actuality, however, it is the *positive* root that must eventually dominate, because as t increases, e^{2t} will grow increasingly larger, but e^{-5t} will steadily dwindle away.

For Case 2, with repeated roots, the complementary function (15.9) contains not only the familiar e^{rt} expression, but also a multiplicative expression te^{rt}. For the former term to approach zero whatever the initial conditions may be, it is necessary-and-sufficient to have $r < 0$. But would that also ensure the vanishing of te^{rt}? As it turns out, the expression te^{rt} (or, more generally, $t^k e^{rt}$) possesses the same general type of time path as does e^{rt} ($r \neq 0$). Thus the condition $r \to 0$ is indeed necessary-and-sufficient for the entire complementary function to approach zero as $t \to \infty$, yielding a dynamically stable intertemporal equilibrium.

EXERCISE 15.1

1 Find the particular integral of each equation:

(a) $y''(t) - 2y'(t) + 5y = 2$
(b) $y''(t) + y'(t) = 7$
(c) $y''(t) + 3y = 9$
(d) $y''(t) + 2y'(t) - y = -4$
(e) $y''(t) = 12$

2 Find the complementary function of each equation:

(a) $y''(t) + 3y'(t) - 4y = 12$
(b) $y''(t) + 6y'(t) + 5y = 10$
(c) $y''(t) - 2y'(t) + y = 3$
(d) $y''(t) + 8y'(t) + 16y = 0$

3 Find the general solution of each differential equation in the preceding problem, and then definitize the solution with the initial conditions $y(0) = 4$ and $y'(0) = 2$.

4 Are the intertemporal equilibriums found in the preceding problem dynamically stable?

5 Verify that the definite solution in Example 5 indeed (a) satisfies the two initial conditions and (b) has first and second derivatives that conform to the given differential equation.

6 Show that, as $t \to \infty$, the limit of te^{rt} is zero if $r < 0$, but is infinite if $r \geq 0$.

An Example of Solution

Let us find the solution of the differential equation

$$y''(t) + 2y'(t) + 17y = 34$$

with the initial conditions $y(0) = 3$ and $y'(0) = 11$.

Since $a_1 = 2$, $a_2 = 17$, and $b = 34$, we can immediately find the particular integral to be

$$y_p = \frac{b}{a_2} = \frac{34}{17} = 2 \qquad \text{[by (15.3)]}$$

Moreover, since $a_1^2 = 4 < 4a_2 = 68$, the characteristic roots will be the pair of conjugate complex numbers $(h \pm vi)$, where

$$h = -\frac{1}{2}a_1 = -1 \qquad \text{and} \qquad v = \frac{1}{2}\sqrt{4a_2 - a_1^2} = \frac{1}{2}\sqrt{54} = 4$$

Hence, by (15.24′), the complementary function is

$$y_c = e^{-t}(A_5 \cos 4t + A_6 \sin 4t)$$

Combining y_c and y_p, the general solution can be expressed as

$$y(t) = e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2$$

To definitize the constants A_5 and A_6, we utilize the two initial conditions. First, by setting $t = 0$ in the general solution, we find that

$$y(0) = e^0(A_5\cos 0 + A_6\sin 0) + 2$$
$$= (A_5 + 0) + 2 = A_5 + 2 \qquad [\cos 0 = 1;\ \sin 0 = 0]$$

By the initial condition $y(0) = 3$, we can thus specify $A_5 = 1$. Next, let us differentiate the general solution with respect to t—using the product rule and the derivative formulas (15.17) and (15.18) while bearing in mind the chain rule [Exercise 15.2-5]—to find $y'(t)$ and then $y'(0)$:

$$y'(t) = -e^{-t}(A_5\cos 4t + A_6\sin 4t) + e^{-t}[A_5(-4\sin 4t) + 4A_6\cos 4t]$$

so that

$$y'(0) = -(A_5\cos 0 + A_6\sin 0) + (-4A_5\sin 0 + 4A_6\cos 0)$$
$$= -(A_5 + 0) + (0 + 4A_6) = 4A_6 - A_5$$

By the second initial condition $y'(0) = 11$, and in view that $A_5 = 1$, it then becomes clear that $A_6 = 3$.* The definite solution is, therefore,

(15.25) $\quad y(t) = e^{-t}(\cos 4t + 3\sin 4t) + 2$

As before, the y_p component ($= 2$) can be interpreted as the intertemporal equilibrium level of y, whereas the y_c component represents the deviation from equilibrium. Because of the presence of circular functions in y_c, the time path (15.25) may be expected to exhibit a fluctuating pattern. But what specific pattern will it involve?

The Time Path

We are familiar with the paths of a simple sine or cosine function, as shown in Fig. 15.4. Now we must study the paths of certain variants and combinations of sine and cosine functions so that we can interpret, in general, the complementary function (15.24′)

$$y_c = e^{ht}(A_5\cos vt + A_6\sin vt)$$

and, in particular, the y_c component of (15.25).

Let us first examine the term $(A_5\cos vt)$. By itself, the expression $(\cos vt)$ is a circular function of (vt), with period 2π ($= 6.2832$) and amplitude 1. The period of 2π means that the graph will repeat its configuration every time that (vt) increases by 2π. When t alone is taken as the independent variable, however, repetition will occur every time t increases by $2\pi/v$, so that with reference to t—as is appropriate in dynamic economic analysis—we shall consider the period of $(\cos vt)$ to be $2\pi/v$. (The amplitude, however, remains at 1.) Now, when a multiplicative constant A_5 is attached to $(\cos vt)$, it causes the range of fluctuation

* Note that, here, A_6 indeed turns out to be a real number, even though we have included the imaginary number i in its definition.

change from ± 1 to $\pm A_5$. Thus the amplitude now becomes A_5, though the period is unaffected by this constant. In short, $(A_5 \cos vt)$ is a cosine function of t, with period $2\pi/v$ and amplitude A_5. By the same token, $(A_6 \sin vt)$ is a sine function of t, with period $2\pi/v$ and amplitude A_6.

There being a common period, the sum $(A_5 \cos vt + A_6 \sin vt)$ will also display a repeating cycle every time t increases by $2\pi/v$. To show this more rigorously, let us note that for given values of A_5 and A_6 we can always find two constants A and ε, such that

$$A_5 = A \cos \varepsilon \qquad \text{and} \qquad A_6 = -A \sin \varepsilon$$

Thus we may express the said sum as

$$\begin{aligned} A_5 \cos vt + A_6 \sin vt &= A \cos \varepsilon \cos vt - A \sin \varepsilon \sin vt \\ &= A(\cos vt \cos \varepsilon - \sin vt \sin \varepsilon) \\ &= A \cos(vt + \varepsilon) \qquad [\text{by (15.16)}] \end{aligned}$$

This is a modified cosine function of t, with amplitude A and period $2\pi/v$, because every time that t increases by $2\pi/v$, $(vt + \varepsilon)$ will increase by 2π, which will complete a cycle on the cosine curve.

Had y_c consisted only of the expression $(A_5 \cos vt + A_6 \sin vt)$, the implication would have been that the time path of y would be a never-ending, constant-amplitude fluctuation around the equilibrium value of y, as represented by y_p. But there is, in fact, also the multiplicative term e^{ht} to consider. This latter term is of major importance, for, as we shall see, it holds the key to the question of whether the time path will converge.

If $h > 0$, the value of e^{ht} will increase continually as t increases. This will produce a magnifying effect on the amplitude of $(A_5 \cos vt + A_6 \sin vt)$ and cause ever-greater deviations from the equilibrium in each successive cycle. As illustrated in Fig. 15.6*a*, the time path will in this case be characterized by *explosive fluctuation*. If $h = 0$, on the other hand, then $e^{ht} = 1$, and the complementary function will simply be $(A_5 \cos vt + A_6 \sin vt)$, which has been shown to have a constant amplitude. In this second case, each cycle will display a uniform pattern of deviation from the equilibrium as illustrated by the time path in Fig. 15.6*b*. This is a time path with *uniform fluctuation*. Last, if $h < 0$, the term e^{ht} will continually decrease as t increases, and each successive cycle will have a smaller amplitude than the preceding one, much as the way a ripple dies down. This case is illustrated in Fig. 15.6*c*, where the time path is characterized by *damped fluctuation*. The solution in (15.25), with $h = -1$, exemplifies this last case. It should be clear that only the case of damped fluctuation can produce a *convergent* time path; in the other two cases, the time path is *nonconvergent* or *divergent*.*

In all three diagrams of Fig. 15.6, the intertemporal equilibrium is assumed to be stationary. If it is a moving one, the three types of time path depicted will still fluctuate around it, but since a moving equilibrium generally plots as a curve

*We shall use the two words *nonconvergent* and *divergent* interchangeably, although the latter is more strictly applicable to the explosive than to the uniform variety of nonconvergence.

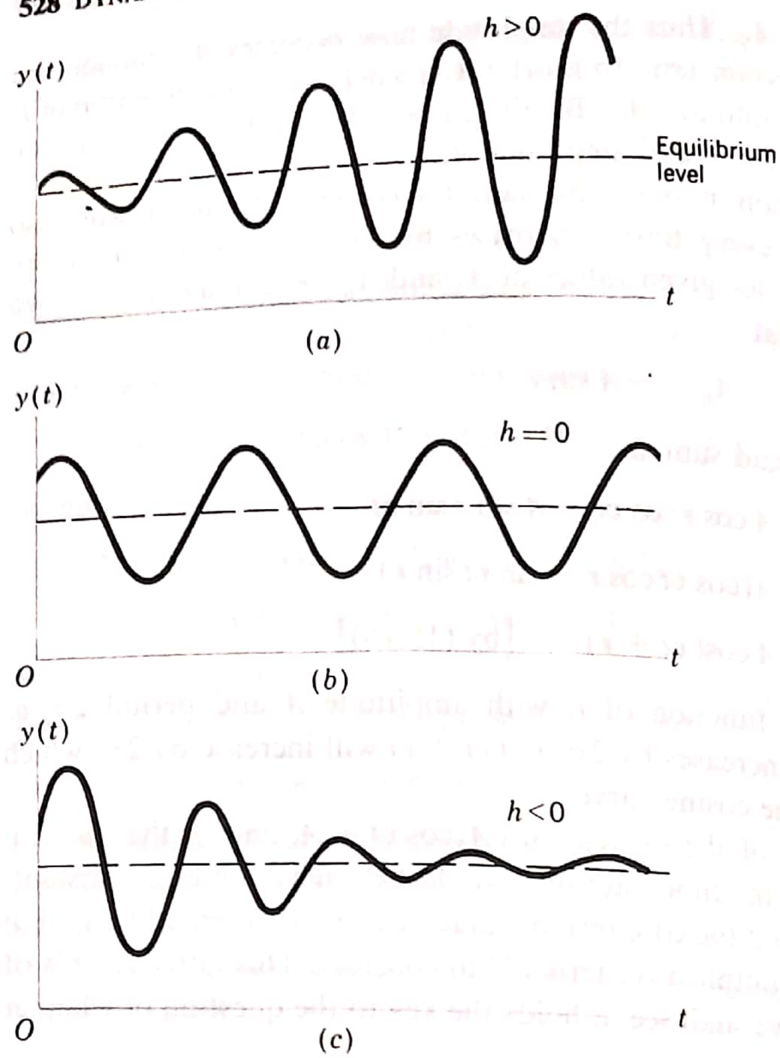

Figure 15.6

rather than a horizontal straight line, the fluctuation will take on the nature of, say, a series of business cycles around a secular trend.

The Dynamic Stability of Equilibrium

The concept of convergence of the time path of a variable is inextricably tied to the concept of dynamic stability of the intertemporal equilibrium of that variable. Specifically, the equilibrium is dynamically stable if, and only if, the time path is convergent. The condition for convergence of the $y(t)$ path, namely, $h < 0$ (Fig. 15.6c), is therefore also the condition for dynamic stability of the equilibrium of y.

You will recall that, for Cases 1 and 2 where the characteristic roots are real, the condition for dynamic stability of equilibrium is that every characteristic root be negative. In the present case (Case 3), with complex roots, the condition seems to be more specialized; it stipulates only that the real part (h) of the complex roots ($h \pm vi$) be negative. However, it is possible to unify all three cases and consolidate the seemingly different conditions into a single, generally applicable one. Just interpret any real root r as a complex root whose imaginary part is zero ($v = 0$). Then the condition "the *real* part of every characteristic root be negative" clearly becomes applicable to all three cases and emerges as the only condition we need.

EXERCISE 15.3

Find the y_p and the y_c, the general solution, and the definite solution of each of the following:

1. $y''(t) - 4y'(t) + 8y = 0$; $y(0) = 3$, $y'(0) = 7$
2. $y''(t) + 4y'(t) + 8y = 2$; $y(0) = 2\frac{1}{4}$, $y'(0) = 4$
3. $y''(t) + 3y'(t) + 4y = 12$; $y(0) = 2$, $y'(0) = 2$
4. $y''(t) - 2y'(t) + 10y = 5$; $y(0) = 6$, $y'(0) = 8\frac{1}{2}$
5. $y''(t) + 9y = 3$; $y(0) = 1$, $y'(0) = 3$
6. $2y''(t) - 12y'(t) + 20y = 40$; $y(0) = 4$, $y'(0) = 5$

7. Which of the above six differential equations yield time paths with (*a*) damped fluctuation; (*b*) uniform fluctuation; (*c*) explosive fluctuation?

... are both negative. ... equilibrium is ensured when the ... gative. In short, ...

Example 1 Let the demand and supply functions be

$$Q_d = 42 - 4P - 4P' + P''$$

$$Q_s = -6 + 8P$$

with initial conditions $P(0) = 6$ and $P'(0) = 4$. Assuming market clearance at every point of time, find the time path $P(t)$.

In this example, the parameter values are

$$\alpha = 42 \qquad \beta = 4 \qquad \gamma = 6 \qquad \delta = 8 \qquad m = -4 \qquad n = 1$$

Since n is positive, our previous discussion suggests that only Case 1 can arise, and that the two (real) roots r_1 and r_2 will take opposite signs. Substitution of the parameter values into (15.28) indeed confirms this, for

$$r_1, r_2 = \frac{1}{2}(4 \pm \sqrt{16 + 48}) = \frac{1}{2}(4 \pm 8) = 6, -2$$

The general solution is, then, by (15.29),

$$P(t) = A_1 e^{6t} + A_2 e^{-2t} + 4$$

By taking the initial conditions into account, moreover, we find that $A_1 = A_2 = 1$, so the definite solution is

$$P(t) = e^{6t} + e^{-2t} + 4$$

In view of the positive root $r_1 = 6$, the intertemporal equilibrium ($\bar{P}_p = 4$) is dynamically unstable.

The above solution is found by use of formulas (15.28) and (15.29). Alternatively, we can first equate the given demand and supply functions to obtain the differential equation

$$P'' - 4P' - 12P = -48$$

and then solve this equation as a specific case of (15.2).

Example 2 Given the demand and supply functions

$$Q_d = 40 - 2P - 2P' - P''$$

$$Q_s = -5 + 3P$$

with $P(0) = 12$ and $P'(0) = 1$, find $P(t)$ on the assumption that the market is always cleared.

Here the parameters m and n are both negative. According to our previous general discussion, therefore, the intertemporal equilibrium should be dynamically stable. To find the specific solution, we may first equate Q_d and Q_s to obtain the differential equation (after multiplying through by -1)

$$P'' + 2P' + 5P = 45$$

The intertemporal equilibrium is given by the particular integral

$$P_p = \frac{45}{5} = 9$$

From the characteristic equation of the differential equation,

$$r^2 + 2r + 5 = 0$$

we find that the roots are complex:

$$r_1, r_2 = \frac{1}{2}(-2 \pm \sqrt{4 - 20}) = \frac{1}{2}(-2 \pm 4i) = -1 \pm 2i$$

This means that $h = -1$ and $v = 2$, so the general solution is

$$P(t) = e^{-t}(A_5 \cos 2t + A_6 \sin 2t) + 9$$

To definitize the arbitrary constants A_5 and A_6, we set $t = 0$ in the general solution, to get

$$P(0) = e^0(A_5 \cos 0 + A_6 \sin 0) + 9 = A_5 + 9 \qquad [\cos 0 = 1; \quad \sin 0 = 0]$$

Moreover, by differentiating the general solution and then setting $t = 0$, we find that

$$P'(t) = -e^{-t}(A_5 \cos 2t + A_6 \sin 2t) + e^{-t}(-2A_5 \sin 2t + 2A_6 \cos 2t)$$

[product rule and chain rule]

and $$P'(0) = -e^{0}(A_5 \cos 0 + A_6 \sin 0) + e^{0}(-2A_5 \sin 0 + 2A_6 \cos 0)$$
$$= -(A_5 + 0) + (0 + 2A_6) = -A_5 + 2A_6$$

Thus, by virtue of the initial conditions $P(0) = 12$ and $P'(0) = 1$, we have $A_5 = 3$ and $A_6 = 2$. Consequently, the definite solution is

$$P(t) = e^{-t}(3 \cos 2t + 2 \sin 2t) + 9$$

This time path is obviously one with periodic fluctuation; the period is $2\pi/v = \pi$. That is, there is a complete cycle every time that t increases by $\pi = 3.14159\ldots$. In view of the multiplicative term e^{-t}, the fluctuation is damped. The time path, which starts from the initial price $P(0) = 12$, converges to the intertemporal equilibrium price $P_p = 9$ in a cyclical fashion.