§ 2.8. CAUCHY SEQUENCE : CAUCHY'S PRINCIPLE OF CONVERGENCE

The following theorem is extremely useful to determine convergence or otherwise of a sequence.

Theorem 2.8.1: A necessary and sufficient condition for the convergence of a sequence $\{x_n\}_n$ is that for a preassigned $\varepsilon\ (>0)$ there exists a positive integer m such that

$|x_{n+p} - x_n| < \varepsilon \quad \forall\ n \geq m$ and for integral values of $p \geq 1$. **(C.H., 1992)**

Proof. The condition is *necessary.*

$\therefore$ Let $\{x_n\}_n$ be convergent to the limit l. Therefore, for a preassigned positive ε it is possible to find a positive integer m such that

$$|x_n - l| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m.$$

Now if $p \geq 1$, $\quad n + p > n \geq m$ and so

$$|x_{n+p} - l| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m \text{ and } p \geq 1.$$

$\therefore\ |x_{n+p} - x_n| = |x_{n+p} - l + l - x_n|$

$\leq |x_{n+p} - l| + |x_n - l| < \epsilon\ \forall\, n \geq m$ and $p \geq 1$.

Sufficiency : We show that under the condition given $\{x_n\}_n$ is bounded and converges to a limit.

Let us choose $\varepsilon = 1$ and $n = m'$. Then from the given condition,

$$|x_{m'+p} - x_{m'}| < 1\ \forall\, p \geq 1$$

i.e., $\qquad x_{m'} - 1 < x_{m'+p} < x_{m'} + 1 \quad \forall\, p \geq 1.$

Let $g = \min(x_1, x_2, \ldots, x_{m'}, x_{m'} - 1)$

$G = \max(x_1, x_2, \ldots x_{m'}, x_{m'} + 1)$

Then $g \leq x_n \leq G\ \forall\, n$ proves that $\{x_n\}_n$ is bounded. Therefore, by theorem 2.6.1 $\{x_n\}_n$ has a limit point, say, l.

We shall now show that $\lim x_n = l$.

By the given condition, for $\varepsilon\,(>0)$ there exists a positive integer m such that

$$|x_{n+p} - x_n| < \frac{\varepsilon}{3} \qquad \forall\, n \geq m \text{ and } p \geq 1.$$

$\therefore\ |x_{m+p} - x_m| < \dfrac{\varepsilon}{3}$ for $p \geq 1$ (take $n = m$) ... (1)

Since l is a limit point there exists a positive integer M such that for $M > m$

$$|x_M - l| < \frac{\varepsilon}{3} \qquad \text{... (2)}$$

Again since $M > m$.

$$|x_M - x_m| < \frac{\varepsilon}{3} \qquad \text{... (3)}$$

$\therefore\ |x_{m+p} - l| = |x_{m+p} - x_m + x_m - x_M + x_M - l|$

$\leq |x_{m+p} - x_m| + |x_M - x_m| + |x_M - l|$

$< \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} = \varepsilon \qquad \forall\, p \geq 1.$

$\therefore\ |x_n - l| < \varepsilon \quad \forall\, n \geq m.$ $\qquad$ or, $\{x_n\}_n$ converges to l.

Example 2.8.1. The sequence $\{x_n\}_n$ where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ is not convergent. **(C.H., 2004)**

***Solution*:** We prove by contradiction. If possible, let $\{x_n\}_n$ be convergent, then for $\epsilon = \frac{1}{2}$ it would be possible to find a positive integer M such that

$$|x_m - x_n| < \frac{1}{2} \quad \forall\ m, n \geq M.$$

For $m = 2n$, we would get

$$|x_{2n} - x_n| < \frac{1}{2} \qquad \forall\ n \geq M \qquad \dots (1)$$

Now $\left| x_{2n} - x_n \right| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right|$

Now $\qquad n + r \leq 2n \qquad$ for $r = 1, 2, \dots n$.

$$\frac{1}{n+r} \geq \frac{1}{2n} \qquad \text{for } r = 1, 2, \dots n$$

$$\therefore |x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}$$

which contradicts (1).

$\therefore$ Our assumption is not correct.

$\therefore$ $\{x_n\}_n$ is not convergent.

Example 2.8.2. The sequence $\{y_n\}_n$ where $y_n = 1 + \frac{1}{2!} + \dots + \frac{1}{n!}$ is convergent.

***Solution*:** We know $n! = 1, 2, 3 \dots n > 2.2 \dots 2 = 2^{n-1}$

$$\therefore \quad \frac{1}{n!} < \frac{1}{2^{n-1}}.$$

Now for $m > n$

$$|y_m - y_n| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!}.$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{m-1}} = \frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{m-n-1}}\right)$$

$$= \frac{1}{2^n} \frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} < \frac{2}{2^n} = \frac{1}{2^{n-1}}.$$

Now $| y_m - y_n | < \varepsilon$ if $\frac{1}{2^{n-1}} < \varepsilon$

i.e., if $2^{n-1} > \frac{1}{\varepsilon}$ or, if $(n-1) \log 2 > \log \frac{1}{\varepsilon}$ or, $n > 1 + \frac{\log \frac{1}{\varepsilon}}{\log 2}$

We choose $n_0 = \left[1 + \frac{\log \frac{1}{\varepsilon}}{\log 2}\right] + 1.$

$\therefore$ For given any $\varepsilon\ (>0)$ it is possible to find n_0 such that

$$| y_m - y_n | < \varepsilon \quad \text{for} \quad m, n > n_0.$$

$\therefore$ By Cauchy's condition $\{y_n\}_n$ is convergent.

Definition (***Cauchy Sequence***) **:** A real sequence $\{x_n\}_n$ is said to be a **Cauchy Sequence** if for every ε (> 0) there exists a positive integer m such that $| x_p - x_q | < \varepsilon$ for all $p, q > m$. **(C.H., 1997, 2001)**

Example 2.8.3. $\left\{\frac{1}{n+1}\right\}_n$ is a Cauchy sequence.

***Solution*:** Let $m > n$ where m and n are positive integers.

Then $\left|\frac{1}{m+1} - \frac{1}{n+1}\right| = \frac{1}{n+1}\left(1 - \frac{n+1}{m+1}\right) < \frac{1}{n+1} < \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$.

We choose $n_0 = \left[\frac{1}{\varepsilon}\right] + 1.$

Then $\left|\frac{1}{m+1} - \frac{1}{n+1}\right| < \varepsilon$ if $n > n_0$ and $m > n$.

Thus $\left\{\frac{1}{n+1}\right\}_n$ is a Cauchy sequence.

Example 2.8.4. $\{(-1)^n\}_n$ is not a Cauchy sequence.

***Solution*:** Let $U_n = (-1)^n$

Then $U_n = -1$ if n is odd

$= 1$ if n is even.

Let us choose $\varepsilon = \frac{1}{2}$, m an even integer and n an odd integer.

Then $| U_m - U_n | = 2 \nless \varepsilon$

$\therefore$ No positive integer n_0 can be found such that

$$| U_m - U_n | < \frac{1}{2} \text{ for } n, m > n_0$$

Example 2.8.5. $\left\{\frac{n}{n+1}\right\}_n$ is a Cauchy sequence. **(C.H., 2001)**

***Solution*:** Let $x_n = \frac{n}{n+1}$, Then for $m > n$

$$\text{Now } | x_m - x_n | = \left|\frac{m}{m+1} - \frac{n}{n+1}\right| = \frac{m-n}{(m+1)(n+1)}$$

$$= \frac{1}{n+1}\left(1 - \frac{n+1}{m+1}\right) < \frac{1}{n+1} < \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}$$

We choose $n_0 = \left[\frac{1}{\varepsilon}\right] + 1$

Then for $m > n > n_0$, $| x_m - x_n | < \varepsilon$ holds for any preassigned ε (> 0)

$\therefore$ $\left\{\frac{n}{n+1}\right\}_n$ is a Cauchy sequence.

Example 2.8.6. Show that $\{2^n\}_n$ is not a Cauchy sequence. **(C.H., 2002)**

***Solution*:** Let $U_n = 2^n$ and we choose $\varepsilon = 1$. For $m > n$

$| U_m - U_n | = | 2^m - 2^n | = 2^n(2^{m-n} - 1) > 2$ and can never be made less than arbitrary positive ε, in whatever way we choose $m > n > n_0$ where n_0 is a positive integer.

$\therefore$ $\{2^n\}_n$ is not a Cauchy sequence.

Example 2.8.7. Prove or disprove : every bounded sequence is a Cauchy sequence. **(C.H., 2003)**

***Solution*:** The statement is not true. We have seen in **Ex. 2.8.4.** above that $\{(-1)^n\}_n$ is not a Cauchy sequence.

But $-1 \le x_n \le 1$ $\forall n \in \mathbb{N}$, where $x_n = (-1)^n$.

Theorem 2.8.2: Every Cauchy sequence is convergent.

Proof. Let $\{U_n\}_n$ be a Cauchy sequence.

By definition, for $\varepsilon = 1$, there exists a positive integer N_0 such that $| U_m - U_n | < 1$ for $m, n \ge N_0$, where m and n are integers.

For $n = N_0$ we have, $|U_m - U_{N_0}| < 1$. for $m \geq N_0$.

$\therefore \quad U_{N_0} - 1 < U_m < U_{N_0} + 1 \quad$ for all $m \geq N_0$.

Let $k = \min\{U_1, U_2, \ldots, U_{N_0-1}, U_{N_0} - 1\}$

and $K = \max\{U_1, U_2, \ldots U_{N_0-1}, U_{N_0} + 1\}$

Then, $k < U_n < K \;\; \forall\, n \in \mathbb{N}$. $\therefore$ $\{U_n\}_n$ is bounded.

(This shows that every Cauchy sequence is bounded.)

By theorem 2.6.1, $\{U_n\}_n$ has a limit point, say l.

By the given condition, for any preassigned $\varepsilon\,(> 0)$, there exists a positive integer m_0 such that

$$|U_m - U_n| < \frac{\varepsilon}{3} \quad \text{for } m, n \geq m_0$$

or, $$|U_m - U_{m_0}| < \frac{\varepsilon}{3} \quad \text{for } m \geq m_0 \qquad \ldots (1)$$

Since l is a limit point, there exists a positive integer $q > m_0$ such that

$$|U_q - l| < \frac{\varepsilon}{3} \qquad \ldots (2)$$

Again since $q > m_0$ by (1)

$$|U_q - U_{m_0}| < \frac{\varepsilon}{3} \qquad \ldots (3)$$

Now $|U_m - l| = |U_m - U_{m_0} + U_{m_0} - U_q + U_q - l|$

$$\leq |U_m - U_{m_0}| + |U_q - U_{m_0}| + |U_q - l|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \;\; \forall\, m \geq m_0.$$

This shows that $\underset{m \to \infty}{\text{Lt}} U_m = l$ or, $\{U_n\}_n$ is convergent.

Since a convergent sequence has a unique limit, $\{U_n\}_n$ being a Cauchy sequence converges to l.

Theorem 2.8.3: Every Convergent sequence is a Cauchy sequence.

Proof. Let $\{x_n\}_n$ be a sequence converging to l.

$\therefore$ For a given $\varepsilon\,(> 0)$, there exists a positive integer M such that

$$|x_n - l| < \frac{\varepsilon}{2} \;\; \forall\, n > M$$

Let us choose $m > M$, then

$$|x_m - l| < \frac{\varepsilon}{2}$$

Now $|x_m - x_n| = |x_m - l + l - x_n| \leq |x_m - l| + |x_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

when $m, n > M$

$\therefore$ $\{x_n\}_n$ is a Cauchy sequence.

Example 2.8.8. Prove that $\{x_n\}_n$ where $x_n = \sum_{r=0}^{n} \frac{1}{r!}$ is a Cauchy sequence.

***Solution*:** In **Ex. 2.8.2.** we have seen that the sequence $\{x_n\}_n$ where

$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ is a convergent sequence. By above Theorem every convergent sequence is a Cauchy sequence

$\therefore$ $\{x_n\}_n$ is a Cauchy sequence.

Example 2.8.9. Prove that the sequence $\{u_n\}_n$ where

$u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + + (-1)^{n-1}\frac{1}{n}$ is a convergent sequence.

(C.H., 1992)

***Solution*:** We shall show that the sequence $\{u_n\}_n$ is a Cauchy sequence and hence it is convergent.

Here for $q > p$ (p and q are integers).

$$|u_q - u_p| = \left|(-1)^p \frac{1}{p+1} + (-1)^{p+1}\frac{1}{p+2} + \ldots + (-1)^{q-1}\frac{1}{q}\right|$$

$$= \frac{1}{p+1} - \left(\frac{1}{p+2} - \frac{1}{p+3}\right) - \left(\frac{1}{p+4} - \frac{1}{p+5}\right) \ldots$$

$$< \frac{1}{p+1} \text{ (each term within bracket is positive)}$$

$\therefore |u_q - u_p| < \frac{1}{p+1} < \varepsilon$ if $p > n_0$ where n_0 is integral part of $\left(\frac{1}{\varepsilon} - 1\right)$.

$\therefore$ We see that $|u_m - u_n| < \varepsilon$ if $m, n > n_0$.

$\therefore$ $\{u_n\}_n$ is a Cauchy sequence.

Definition : The sequence $\{I_n\}_n$ of closed intervals such that $I_n \supset I_{n+1}$ is called a sequence of **nested intervals.**

We now prove the following important theorem on **Nested intervals.**

Theorem 2.8.4: If $\{I_n\}_n$ be a sequence of non-empty closed intervals such that $I_n \supset I_{n+1}$, then $\bigcap_{n \in \mathbb{N}} I_n$ contains at least one point ξ. If further $\lim_{n \to \infty} |I_n| = 0$, then ξ is unique. ($|I_n|$ denotes the length of I_n)

(C.H., 1997, 2003)

Proof. Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ and so on $I_n = [a_n, b_n]$ etc.

Since $I_n \supset I_{n+1}$, we have

$a_1 \le a_2 \le ... \le a_n \le .... < b_{n+1} \le b_n \le ... \le b_1$.

We thus get two sequences of real numbers $\{a_n\}_n$ and $\{b_n\}_n$ of which

(i) $\{a_n\}_n$ is a monotone increasing sequence and bounded above by b_1 (in fact by each b_n).

(ii) $\{b_n\}_n$ is a monotone decreasing sequence and bounded below by a_1 (in fact by each a_n). Hence both the sequences $\{a_n\}_n$ and $\{b_n\}_n$ are convergent.

Let $\lim_{n \to \infty} a_n = x$ and $\lim_{n \to \infty} b_n = y$.

x = l.u.b. of $\{a_i, i \in \mathbb{N}\}$ and let, if possible $b_m < x$ for some $m \in \mathbb{N}$.

Then $b_m < a_r < x \Rightarrow b_k \le b_m < a_r \le a_k$ where $\max(m, r) = k$ which is impossible as $[a_k, b_k]$ is an interval.

$\therefore\ x \le b_n\ \forall\ n \in \mathbb{N}$. $\quad \therefore \bigcap_{n \in \mathbb{N}} I_n$ contains at least one point.

Now $|I_n| = b_n - a_n$ and $\lim_{n \to \infty} |I_n| = 0$ implies that $\lim_{n \to \infty} (b_n - a_n) = 0$

or, $x - y = 0$, $\quad$ or, $x = y$.

Hence $a_n \le x \le b_n$. $\quad \therefore x \in I_n\ \forall\ n \in \mathbb{N}$

If possible let x_1 and x_2 $(x_1 < x_2)$ be two different points such that

$x_1 \in \bigcap_{n \in \mathbb{N}} I_n$ and $x_2 \in \bigcap_{n \in \mathbb{N}} I_n$. $\quad \therefore a_n \le x_1 < x_2 \le b_n$

$\therefore\ b_n - a_n \ge x_2 - x_1\ \forall n$ which contradicts that $(b_n - a_n) \to 0$ as $n \to \infty$

$\therefore x_1$ is not different from x_2

Hence $\bigcap_{n \in \mathbb{N}} I_n$ is a unique point.

Note : The theorem may fail when the intervals are not closed. **(C.H., 1997, 2003)**

Let us choose $I_n = \left(0, \frac{1}{n}\right)$ such that $|I_n| = \frac{1}{n}$ which tends to 0 as n tends to infinity. Further $I_n \supset I_{n+1}$. But we see that there is no point ξ such that $\xi \in \bigcap_{n \in \mathbb{N}} I_n$.

In fact, $\bigcap_{n \in \mathbb{N}} I_n = \phi$.

1. Show that the sequence $\{a_n\}_n$ where

$$a_n = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \dots + \frac{1}{(2n-1)(2n+1)}$$ is monotone increasing and bounded.

Solution: $a_n = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}.$$

$a_{n+1} = \frac{n+1}{2n+3}$ $\quad \therefore a_{n+1} - a_n = \frac{n+1}{2n+3} - \frac{n}{2n+1} = \frac{1}{(2n+1)(2n+3)} > 0.$

$\therefore a_{n+1} > a_n \ \forall n \in \mathbb{N}$ $\quad \therefore \{a_n\}_n$ is monotone increasing.

Now, $0 < \frac{n}{2n+1} < 1$ $\quad \therefore \{a_n\}_n$ is bounded.

2. Show that the sequence $\{a_n\}_n$ where

$$a_n = \frac{1}{n}\cos\frac{n\pi}{2}$$ is convergent.

Solution: Here $|a_n - 0| = \left|\frac{1}{n}\cos\frac{n\pi}{2}\right| \le \frac{1}{n} < \varepsilon$

if $n > \frac{1}{\varepsilon}$ we choose $m = \left[\frac{1}{\varepsilon}\right] + 1.$

$\therefore |a_n - 0| < \varepsilon$ for $n > m.$ $\quad \therefore \{a_n\}_n$ converges to 0.

3. Prove that $\lim\limits_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0.$ **(B.H., 2002)**

Solution: Here $\left|\sqrt{n+1} - \sqrt{n} - 0\right| = \dfrac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}}$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \qquad \left(\because \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}\right)$$

$\therefore \left|\sqrt{n+1} - \sqrt{n} - 0\right| < \epsilon$, if $\frac{1}{2\sqrt{n}} < \varepsilon$ or, if $n > \frac{1}{4\varepsilon^2}$

We choose $m = \left[\frac{1}{4\varepsilon^2}\right] + 1$

$\therefore \left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon$ when $n > m.$ $\quad \therefore \lim\limits_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0.$

4. If $u_n = \frac{2}{3} \cdot \frac{5}{7} \cdots \frac{3n-1}{4n-1}$. Show that $\lim_{n \to \infty} u_n = 0$.

Solution: Here $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \frac{3n+2}{4n+3} = \frac{3}{4} < 1$. $\therefore \lim_{n \to \infty} u_n = 0$

[We use the result of the following theorem :

If $\{x_n\}_n$ be a sequence such that

$\lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = l \ (0 \le l < 1)$, then $\lim_{n \to \infty} x_n = 0$]

5. If $u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$ and $v_n = \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots 2n}$,

then show that $\lim_{n \to \infty} u_n = 0$ and $v_n \to \infty$ and $\frac{1}{2} < u_n v_n < 1$.

Solution: Here $\frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5}$, etc. $\frac{2n-1}{2n} < \frac{2n}{2n+1}$.

$$\text{Now, } u_n^2 = \left(\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}\right)\left(\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}\right)$$

$$< \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \times \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1} = \frac{1}{2n+1}$$

$$\therefore u_n < \frac{1}{\sqrt{2n+1}} < \varepsilon, \text{ if } 2n+1 > \frac{1}{\varepsilon^2} \quad \text{or, if } n > \frac{1}{2}\left(\frac{1}{\varepsilon^2} - 1\right)$$

$$\therefore |u_n - 0| < \varepsilon, \text{ if } n > m \text{ where } m = \left[\frac{1}{2}\left(\frac{1}{\varepsilon^2} - 1\right)\right] + 1. \quad \therefore \text{Lt}_{n \to \infty} u_n = 0.$$

$$\text{Again, } \frac{3}{2} > \frac{4}{3}, \frac{5}{4} > \frac{6}{5} \cdots \frac{2n+1}{2n} > \frac{2n+2}{2n+1}$$

$$v_n^2 = \left(\frac{2}{3} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n}\right)^2 > \frac{2}{3} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n+2}{2n+1} = n+1$$

$\therefore$ $v_n > \sqrt{n+1} > G$, if $n > G^2 - 1$ (G is large at pleasure)

$\therefore$ $v_n \to +\infty$ as $n \to \infty$.

$$\text{Now } u_n v_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n}.$$

$$< \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n} = 1 \qquad \ldots (1)$$

Also $u_n v_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n}$

$$> \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n}{2n-1} \cdot \frac{2n+2}{2n+1} = \frac{1}{2} \cdot \frac{2n+2}{2n+1} > \frac{1}{2} \qquad \ldots (2)$$

Combining the two results (1) and (2) $\therefore \frac{1}{2} < u_n v_n < 1.$

6. Given that $\{a_n\}_n$ as a sequence such that $a_2 \le a_4 \le a_6 \le \ldots \le a_5 \le a_3 \le a_1$ and a sequence $\{b_n\}_n$ where $b_n = a_{2n-1} - a_{2n}$ converges to 0, then show that the sequence $\{a_n\}_n$ is convergent.

Solution: $\{a_n\}_n$ consists of two subsequence $\{a_{2n}\}_n$ and $\{a_{2n-1}\}_n$ of which $\{a_{2n}\}_n$ is monotone increasing and $\{a_{2n-1}\}_n$ is monotone decreasing. The sequence $\{a_{2n}\}_n$ is bounded above by a_1 and the sequence $\{a_{2n-1}\}_n$ is bounded below by a_2. Hence both the sequences are convergent.

Let $\{a_{2n}\}_n$ converge to l and $\{a_{2n-1}\}_n$ converge to l'.

Now $\{b_n\}_n$ converges to 0.

$\therefore \lim_{n\to\infty} b_n = 0.$ or, $\lim_{n\to\infty} (a_{2n-1} - a_{2n}) = 0$ or, $l^1 - l = 0.$ or, $l = l^1$

$\therefore$ The sequence $\{a_n\}_n$ converges to l.

7. Let a sequence $\{s_n\}_n$ be defined as $s_{n+1} = \frac{4+3s_n}{3+2s_n}$ $n \ge 1$, $s_1 = 1$.

Show that $\{s_n\}_n$ converges to $\sqrt{2}$.

Solution: Here $s_{n+2} - s_{n+1} = \frac{(s_{n+1} - s_n)}{(3+2s_{n+1})(3+2s_n)}$

$\therefore s_{n+2} > s_{n+1}$ if $s_{n+1} > s_n$, *i.e.*, according as $s_2 > s_1$

Now $s_2 = \frac{7}{5} > s_1$ $\therefore \{s_n\}_n$ is a monotone increasing sequence.

Now, $s_{n+1} - 1 = \frac{1+s_n}{3+2s_n} < 1$

$\therefore 0 < s_n < 2 \;\forall n.$ $\therefore \{s_n\}_n$ is a convergent sequence.

Let $\lim_{n\to\infty} s_n = l.$ $\therefore \lim_{n\to\infty} s_{n+1} = \frac{4+3\lim_{n\to\infty} s_n}{3+2\lim_{n\to\infty} s_n}$

$$l = \frac{4+3l}{3+2l} \qquad \therefore\ l^2 = 2 \qquad \therefore\ l = \sqrt{2}$$

Since the terms are all positive, l can not be negative.

8. A sequence $\{x_n\}_n$ is defined as follows

$$x_{n+1} = \sqrt{\frac{ab^2 + x_n^2}{a+1}} \quad \forall\ n \geq 1 \text{ and } x_1 = a > 0.$$

Prove that (i) $\{x_n\}_n$ is monotone decreasing and bounded if $x_1 > b$.

(ii) $\{x_n\}_n$ is monotone increasing and bounded if $x_1 < b$.

(iii) in either case $\{x_n\}_n$ converges to b.

Solution: $x_{n+1}^2 - x_n^2 = \dfrac{x_n^2 - x_{n-1}^2}{a+1} = \dots = \dfrac{x_2^2 - x_1^2}{(a+1)^{n-1}} = \dfrac{a\left(b^2 - x_1^2\right)}{(a+1)^n}$

$\therefore\ x_{n+1} \gtreqless x_n$ according as $b \gtreqless x_1$

$\therefore\ \{x_n\}_n$ is monotone increasing or decreasing according as $x_1 < b$ or, $x_1 > b$.

Now, $x_{n+1}^2 - b^2 = \dfrac{x_n^2 - b^2}{a+1} = \dfrac{x_{n-1}^2 - b^2}{(a+1)^2} = \dots = \dfrac{x_1^2 - b^2}{(a+1)^n}$.

if $x_1 < b$, $0 < x_n < b$, then $\{x_n\}_n$ is monotone increasing and bounded above.

if $x_1 > b$ $x_n > b$ and then $\{x_n\}_n$ is monotone decreasing and bounded below,

In either case $\{x_n\}_n$ is convergent.

Let $\lim\limits_{n \to \infty} x_n = l$. $\therefore$ Taking limit we have, $x_{n+1} \to \sqrt{\dfrac{ab^2 + x_n^2}{a+1}}$ as $n \to \infty$

$$\Rightarrow\ l^2 = \frac{ab^2 + l^2}{a+1} \qquad \therefore\ l = b.$$

$\therefore\ \{x_n\}_n$ converges to b.

9. If the sequence $\{a_n\}_n$ and $\{b_n\}_n$ converge to A and B respectively, then $\lim \frac{1}{n}\,(a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1) = \text{AB}$. **(B.H., 2000)**

Solution: Let us put $a_n = \text{A} + x_n$ and $|x_n| = \text{X}_n$.

Since $\{a_n\}_n$ converges to A, $\lim a_n = \text{A}$ $\therefore\ x_n \to 0$ and hence $\text{X}_n \to 0$

$\therefore$ By Cauchy's first theorem on limit $\lim\limits_{n \to \infty} \dfrac{(\text{X}_1 + \dots + \text{X}_n)}{n} = 0$ ... (1)

Now $\frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1)$

$$= \frac{1}{n}[(A + x_1)b_n + (A + x_2) b_{n-1} + \cdots + (A + x_n)b_1]$$

$$= \frac{A}{n}(b_1 + \cdots + b_n) + \frac{1}{n}(x_1 b_n + \cdots + x_n b_1) \quad \ldots (2)$$

$\{b_n\}_n$ converges implies $\{b_n\}_n$ is bounded.

Hence $|b_n| < k$ for all n.

$$\therefore \quad \frac{1}{n}|(x_1 b_n + \cdots + x_n b_1)| < \frac{k}{n}|x_1| + \cdots + |x_n| = \frac{k}{n}(X_1 + \cdots + X_n)$$

$$\therefore \quad \lim_{n\to\infty} \frac{1}{n}|(x_1 b_n + \cdots + x_n b_1)| = 0 \quad \text{by (1)}$$

Again $\lim_{n\to\infty} b_n = B.$

$$\therefore \quad \lim_{n\to\infty} \frac{1}{n}(b_1 + \cdots + b_n) = B \text{ [By Cauchy's first limit theorem].}$$

$$\therefore \text{ By (2) } \lim \frac{1}{n}(a_1 b_{n-1} + a_2 b_n + \ldots + a_n b_1) = AB.$$

10. Show by Cauchy's general principle of convergence that the sequence $\left\{\frac{n-1}{n+1}\right\}_n$ is convergent. **(C.H., 1983)**

Solution: Here $x_n = \frac{n-1}{n+1}$. we take positive integers m and n such that $m > n$.

$$|x_m - x_n| = \left|\frac{m-1}{m+1} - \frac{n-1}{n+1}\right| = \frac{2(m-n)}{(m+1)(n+1)}$$

$$< \frac{2}{n}\frac{\left(1 - \frac{n}{m}\right)}{(n+1)^2} < \frac{2}{n^3} < \varepsilon \text{ if } n^3 > \frac{2}{\varepsilon} \text{ or } n > \left(\frac{2}{\varepsilon}\right)^{\frac{1}{3}}.$$

We choose $n_0 = \left[\frac{2}{\varepsilon}\right]^{\frac{1}{3}} + 1$. Then $|x_m - x_n| < \varepsilon \ \forall\ m, n > n_0$.

Hence $\{x_n\}_n$ is convergent.

11. Prove that the sequences $\{x_n\}_n$ and $\{y_n\}_n$ where $x_n = \sum_{r=1}^{n}\frac{1}{r} - \log n$ and $y_n = \sum_{r=1}^{n-1}\frac{1}{r} - \log n$ $(n \geq 2)$ converge to the same limit.

Solution: Here, $x_{n+1} - x_n = \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right)$

$$y_{n+1} - y_n = \frac{1}{n} - \log\left(1+\frac{1}{n}\right)$$

$\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}_n$ is strictly monotone decreasing and bounded below, therefore convergent and converges to e.

Also, $\left\{\left(1+\frac{1}{n}\right)^{n}\right\}_n$ is strictly monotone increasing and bounded above, therefore convergent and converges to e.

$$\therefore \quad \left(1+\frac{1}{n}\right)^{n+1} > e \Rightarrow \log\left(1+\frac{1}{n}\right) > \frac{1}{n+1} \ \forall\, n \qquad \text{... (1)}$$

and
$$\left(1+\frac{1}{n}\right)^{n} < e \Rightarrow \log\left(1+\frac{1}{n}\right) < \frac{1}{n} \qquad \text{... (2)}$$

$\therefore \{x_n\}_n$ is monotone decreasing and $\{y_n\}_n$ is monotone increasing.

(2) $\Rightarrow \frac{1}{n} > \log\,(n+1) - \log n \Rightarrow \Sigma\frac{1}{n} > \log\,(n+1) > \log n$

$\therefore x_n > 0\ \forall n$ shows that $\{x_n\}_n$ is monotone decreasing and bounded below, hence convergent.

Let $\lim\limits_{n\to\infty} x_n = \gamma$

Also $\lim\,(x_n - y_n) = \lim \frac{1}{n} = 0 \quad \therefore \{y_n\}_n$ converges to γ.

Note : $x_1 = 1$ and since $\{x_n\}_n$ is monotone decreasing, $\gamma < 1$

$y_2 = 1 - \log 2 > 0.3 \quad \{y_n\}_n$ being monotone increasing.

$\gamma > 0.3$. Hence, $0.3 < \gamma < 1$. γ is called **Euler's constant.**

5. Examine whether the following sequences are Cauchy sequences or not.

(a) $\left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}_n$ (b) $\left\{1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right\}_n$

(c) $\{(-1)^n_n\}_n$ (d) $\left\{\frac{n-1}{n+1}\right\}_n$ (e) $\left\{\frac{1}{n}\right\}_n$ **(C.H., 1988)**