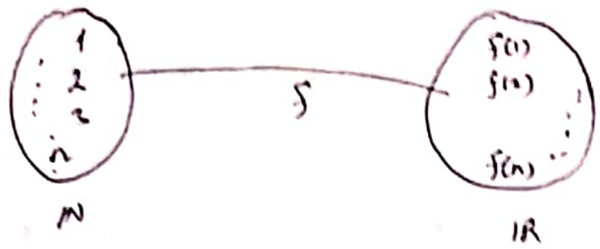
Sequence of Real Numbers

Defn: A mapping $f: \mathbb{N} \to \mathbb{R}$ is said to be a sequence of real number or a real sequence. and it is denoted by $\{f(n)\}$



or $\{x_n\}$ or $\{a_n\}$. Here x_n or a_n or $f(n)$ is the nth term of the sequence.

Ex:

① $x_n = \frac{1}{n}$ Then $\{\frac{1}{n}\}$ is a Sequence, i.e, $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is a seq.

② $a_n = n$. Then $\{n\}$ is a sequence, i.e, $\{1, 2, \cdots\}$ " "

③ $x_n = (-1)^n$. Then $\{(-1)^n\}$ is a Seq. i.e, $\{-1, 1, -1, 1, \cdots\}$ is a Seq.

So, the range set of seq $\{\frac{1}{n}\}$ is $\{1, \frac{1}{2}, \frac{1}{3} \cdots\}$

" " $\{n\}$ is $\{1, 2, 3, \cdots\}$

" " $\{(-1)^n\}$ is $\{-1, 1\}$.

So, from above, it is clear that the range set of a seq of real no may be finite or infinite.

Some Definitions:

Bounded above sequence.

A real sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real no M_1 s.t $a_n \le M_1$ $\forall n \in \mathbb{N}$. M_1 is said to be an upper bound of the seq $\{a_n\}$.

Bounded below Sequence.

A real sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real no m_1 s.t $a_n \ge m_1$ $\forall n \in \mathbb{N}$. m_1 is said to be a lower bound of $\{a_n\}$.

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Bounded sequence:

A real seq $\{a_n\}$ is said to be bounded if it is both bounded above and bounded below. ie, $\exists$ two real nos m_1 and M_1 s.t

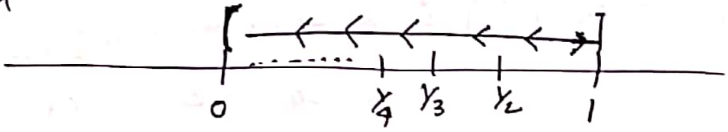
$$m_1 \leq a_n \leq M_1 \quad \forall n \in \mathbb{N}.$$

or, $\exists$ a real no M s.t $\quad |a_n| \leq M \quad \forall n \in \mathbb{N}.$

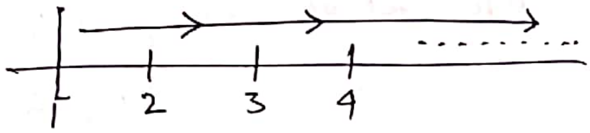
$$\Rightarrow \underbrace{-M}_{\text{lower bound}} \leq a_n \leq \underbrace{M}_{\text{upper bound.}}$$

Examples:

① The seq $\{\frac{1}{n}\}$ is bounded. 0 is a lower bound and 1 is an upper bound



② The seq $\{n\}$ is a bounded below seq not bounded above. 1 being an lower bound.



③ The seq $\{(-1)^n n\}$ is not either bounded above or bounded below.

④ The seq $\{(-1)^n\}$ is a bounded seq.

Limit of a sequence:

Let $\{a_n\}$ be a real sequence. A real number l is said to be a limit of the seq $\{a_n\}$ if for given $\varepsilon > 0$ $\exists$ a Positive integer $K \in \mathbb{N}$ s.t

$$|a_n - l| < \varepsilon \quad \forall n \geq K.$$

$$\Rightarrow -\varepsilon < a_n - l < \varepsilon \quad \forall n \geq K.$$

$$\Rightarrow l - \varepsilon < a_n < l + \varepsilon \quad \forall n \geq K.$$

$$\Rightarrow a_n \in (l-\varepsilon, l+\varepsilon) \quad \forall n \geq K \text{ ie, } a_K, a_{K+1}, a_{K+2}, \ldots \in (l-\varepsilon, l+\varepsilon)$$

i.e. all the elements of $\{a_n\}$, except the first $K-1$ terms, namely $a_1, a_2, \ldots a_{K-1}$ lies in the neighbourhood $(l-\varepsilon, l+\varepsilon)$.

Note: In the above definition K depends on ε. If we change the value of ε, then K will be changed. For example, consider the sequence $\{\frac{1}{n}\}$.

i.e. $\{ \underset{a_1}{1}, \underset{a_2}{1/2}, \underset{a_3}{1/3}, \underset{a_4}{1/4}, \underset{a_5}{1/5}, \underset{a_6}{1/6}, \underset{a_7}{\frac{1}{7}}, \ldots \}$

Let $\varepsilon = 1/3$, Then

$$\left.\begin{array}{l} |a_1 - 0| = 1 \nless 1/3 \\ |a_2 - 0| = 1/2 \nless 1/3 \\ |a_3 - 0| = 1/3 \nless 1/3 \\ |a_4 - 0| = \frac{1}{4} < 1/3 \\ |a_5 - 0| = \frac{1}{5} < 1/3 \end{array}\right\}$$

i.e. for $\varepsilon = 1/3$, the result $|a_n - 0| < 1/3 = \varepsilon$ holds ~~for~~ from the 4th member of the sequence.

So, Here we choose $K = 4$.

and so on.

Again if $\varepsilon = 1/4$, then $|a_n - 0| < 1/4 = \varepsilon$ holds from the 5th term of the Sequence. Here $K = 5$.

i.e. First ε then K. i.e. K depends on ε not conversely.

Theorem: A sequence can have atmost a limit.

or ~~The ordinality of~~

Pf: Let $\{a_n\}$ be a sequence of real nos, have two limits say l_1 and l_2 (say)

Let $\varepsilon = \frac{|l_1 - l_2|}{4} > 0$.

Since l_1 is a limit of $\{a_n\}$, then for given $\varepsilon > 0$ $\exists$ $K_1 \in \mathbb{N}$ s.t

$$|a_n - l_1| < \varepsilon \quad \forall n \geq K_1 \text{ ——— } (1)$$

Again, since l_2 is a limit of $\{a_n\}$, then for above $\varepsilon > 0$, $\exists$ $K_2 \in \mathbb{N}$ s.t

$$|a_n - l_2| < \varepsilon \quad \forall n \geq K_2 \text{ ——— } (2)$$

So, for $n \geq K = \max(K_1, K_2)$ both the results ① and ② holds simultaneously.

Now, $|l_1 - l_2| = |(a_n - l_2) - (a_n - l_1)|$

$$\leq |a_n - l_2| + |a_n - l_1|$$

$$< \varepsilon + \varepsilon \quad \forall n \geq K$$

$\Rightarrow |l_1 - l_2| < 2\varepsilon \Rightarrow 4\varepsilon < 2\varepsilon$, which is absurd as $\varepsilon > 0$ be arbitrary.

So, ~~or~~ $l_1 = l_2$. (Proved).

Convergent Sequence:

A real seq $\{a_n\}$ is said to be a convergent sequence if it has a limit $l \in \mathbb{R}$. and we write $\lim_{n \to \infty} a_n = l$.

A seq $\{a_n\}$ is said to be divergent if it is not convergent.

Example

1) The seq $\{\frac{1}{n}\}$ converges to 0.

Pf: Let $\varepsilon > 0$ be given.

By Archimedean Prop of $\mathbb{R}$, $\exists\ K \in \mathbb{N}$ s.t $0 < \frac{1}{K} < \varepsilon$.

$\Rightarrow 0 < \frac{1}{n} < \varepsilon \quad \forall n \geq K$

$\Rightarrow -\varepsilon < 0 < \frac{1}{n} < \varepsilon \quad \forall n \geq K$

$\Rightarrow 0 - \varepsilon < \frac{1}{n} < 0 + \varepsilon \quad \forall n \geq K$

$\Rightarrow |\frac{1}{n} - 0| < \varepsilon \quad \forall n \geq K. \Rightarrow \lim_{n \to \infty} \frac{1}{n} = 0$

2) The seq $\{a_n\} = \{3, 3, 3, \ldots\ldots\}$ is a constant seq and it converges to 3.

as $|a_n - 3| = 0 < \varepsilon \quad \forall n$

ie, $\lim_{n \to \infty} a_n = 3$

Q: A Constant seq is Convergent, but the converse may not be true.

Th: A Convergent sequence is bounded.

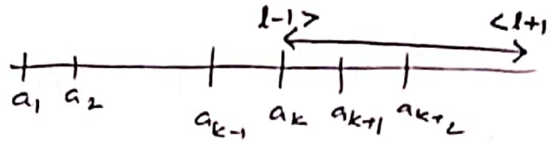
Pf: Let $\{a_n\}$ be a conv^t seq of real nos and it converges to l (say). Then for $\varepsilon = 1 > 0$, $\exists\, K \in \mathbb{N}$ s.t

$$|a_n - l| < 1 = \varepsilon \quad \forall\, n \geq K$$

$$\Rightarrow \quad l-1 < a_n < l+1 \quad \forall\, n \geq K$$

$$\Rightarrow \quad l-1 < a_K, a_{K+1}, a_{K+2}, \cdots\cdots - < l+1$$

let
$$M = \max\{a_1, a_2, \cdots\cdots a_{K-1}, l+1\}$$
$$m = \min\{a_1, a_2, \cdots\cdots a_{K-1}, l-1\}.$$



Then
$$m \leq a_n \leq M \quad \forall\, n \in \mathbb{N}.$$

So, $\{a_n\}$ is bounded.

Q: Is the Converse true? Justify your answer.

Ans: NO. A bounded seq may not be conv^t.

Example: $\{(-1)^n\}$ is a bounded seq but not Convergent.

Limit Theorems:

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences of real numbers that converges to a and b respectively. Then

1) $\lim (a_n + b_n) = a+b$
2) $\lim (c\, a_n) = ca$ for $c \in \mathbb{R}$.
3) $\lim (a_n b_n) = ab$
4) $\lim \dfrac{a_n}{b_n} = \dfrac{a}{b}$, Provided $\{b_n\}$ is a non zero seq and $b \neq 0$

Proof: left as an excercise.

Th: Sandwich Theorem:

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequence of real numbers and there is a natural number k such that

$$a_n \leq b_n \leq c_n \quad \forall\, n \geq k$$

If $\lim a_n = l$, $\lim c_n = l$ then $\lim b_n = l$.

Proof: Since $\{a_n\}, \{c_n\}$ Converges to l then for $\varepsilon > 0$, $\exists\, K_1, K_2 \in \mathbb{N}$ s.t

$$|a_n - l| < \varepsilon/2 \quad \forall\, n \geq K_1 \quad \text{---(1)}$$

$$\& \quad |c_n - l| < \varepsilon/2 \quad \forall\, n \geq K_2 \quad \text{---(2)}$$

Then for $n \geq p = \max(K_1, K_2)$, both (1) and (2) holds Simultaneously.

Let $s = \max(p, k)$. Then for $n \geq s$,

$$l - \varepsilon/2 \leq a_n \leq b_n \leq c_n < l + \varepsilon/2 \quad \forall\, n \geq s$$

$$\Rightarrow |b_n - l| < \varepsilon/2 \quad \forall\, n \geq s.$$

so $\lim b_n = l$. (Proved).

Example: Use Sandwich Theorem to show that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

Sol^n: Let $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

We have $\frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$$

$$\cdots\cdots\cdots$$

$$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}$$

Then $a_n < \frac{n}{\sqrt{n^2+1}} \quad \forall\, n \geq 2$.

Again $\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$

$$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$$

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$$u_n > \frac{n}{\sqrt{n^2+n}} \quad \forall\, n \geq 2.$$

Thus $\frac{n}{\sqrt{n^2+n}} < u_n < \frac{n}{\sqrt{n^2+1}} \quad \forall\, n \geq 2.$

Since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$ & $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$, Then by Sandwich Th,

$\lim_{n\to\infty} u_n = 1.$ (Proved)

Exercise: Use sandwich theorem to show

a) $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$, b) $\lim_{n\to\infty} \left(2^n + 3^n\right)^{1/n} = 3$

c) $\lim_{n\to\infty} \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$

d) $\lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} = 0$

Null Sequence:

A Sequence $\{a_n\}$ is said to be a null Sequence if $\lim_{n\to\infty} a_n = 0$.

Ex: $\{\frac{1}{n}\}$ is a null seq as $\lim_{n\to\infty} \frac{1}{n} = 0$.

Problem: If $\{a_n\}$ is a null Seq, then $\{|a_n|\}$ is also a null Seq and Conversly.

Sol^n: Let $\varepsilon > 0$. Since $\{a_n\}$ is a null Seq, $\lim_{n\to\infty} a_n = 0$. Then for $\varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t. $|a_n - 0| < \varepsilon \;\forall\, n \geq K$.

Now, $||a_n| - 0| = |a_n| < \varepsilon \;\forall\, n \geq K$. so $\lim_{n\to\infty} |a_n| = 0$.

Conversly, let $\lim |a_n| = 0$, Then for $\varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t $||a_n| - 0| < \varepsilon \;\forall n \geq K$.

$\Rightarrow |a_n - 0| < \varepsilon \;\forall\, n \geq K \Rightarrow \lim a_n = 0.$ (Proved)

Divergent sequence:

Divergent to $+\infty$:- A real seq $\{a_n\}$ is said to be diverge to $+\infty$ if for any given +ve real no G (however large it may be), $\exists$ a, $K \in \mathbb{N}$ s.t

$$a_n > G \quad \forall\, n \geq K.$$

and in this Case we write $\lim_{n\to\infty} a_n = +\infty$.

Ex: sequence $\{n\}$, $\{n^2\}$ diverges to $+\infty$.

Divergent to $-\infty$:→ A real seq $\{a_n\}$ is said to diverge to $-\infty$ if for any +ve real no G (however large it may be), $\exists$ a, $K \in \mathbb{N}$ s.t

$$a_n < -G \quad \forall\, n \geq K.$$

and in this Case, we write $\lim_{n\to\infty} a_n = -\infty$.

Ex: $\{-n^3\}$ diverges to $-\infty$

A real sequence $\{a_n\}$ is said to be a properly divergent if it either diverge to $+\infty$ or $-\infty$.

Oscillatory sequence:

Defn: A bounded seq that is not Convt. is said to be an oscillatory sequence of finite oscillation.

Ex: $\{(-1)^n\}$ is a bounded seq which is not Convt, is an oscillatory seq of finite oscillation.

Defn: An unbounded seq that is not properly divergent is said to be an oscillatory seq of infinite oscillation.

Ex: $\{(-1)^n n\}$ is an unbounded seq and it is not properly divergent, It is an oscillatory Seq of infinite oscillation.

Some important Useful limits:

① $\lim_{n\to\infty} r^n = 0$ if $|r|<1$

② $\lim_{n\to\infty} a^{1/n} = 1$ if $a>0$

③ If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} \log(1+x_n) = 0$.

④ $\lim_{n\to\infty} n^{1/n} = 1$.

⑤ If $\{a_n\}$ be a sequence of positive real numbers such that

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

i) If $0 \le l < 1$ then $\lim_{n\to\infty} a_n = 0$

ii) If $l > 1$, then $\lim_{n\to\infty} a_n = +\infty$.

⑥ If $\{a_n\}$ be a sequence of Positive real numbers such that

$\lim_{n\to\infty} a_n^{1/n} = l$, then

i) If $0 \le l < 1$, then $\lim_{n\to\infty} a_n = 0$

ii) If $l > 1$, then $\lim_{n\to\infty} a_n = +\infty$.

Theorem:

1) A sequence diverging to $+\infty$ is unbounded above but bounded below.

Is the Converse true? Justify your answer.

2) A sequence diverging to $-\infty$ is unbounded below but bounded above.

Is the Converse true? justify your answer.

$\S$: MONOTONE SEQUENCE

Monolónic increasing sequence: A real sequence $\{a_n\}$ is said to be monolónic increasing sequence if $a_{n+1} \geq a_n \;\forall\, n \in \mathbb{N}$.

Ex: $\{2^n\}$ is m.i. Sequence.

Monolónic decreasing seq: A real seq $\{a_n\}$ is said to be monolonic decreasing seq if $a_{n+1} \leq a_n \;\forall\, n \in \mathbb{N}$.

Ex: $\{\frac{1}{n}\}$ is a m.d. seq.

strictly increasing sequence: A real seq $\{a_n\}$ is said to be strictly increasing seq if $a_{n+1} > a_n \;\forall\, n \in \mathbb{N}$.

Strictly decreasing sequence: A real seq $\{a_n\}$ is said to be strictly decreasing seq if $a_{n+1} < a_n \;\forall\, n \in \mathbb{N}$.

Ex: $a_n = \dfrac{n}{n+1}$

$$a_{n+1} = \frac{n+1}{n+2}, \quad a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= \frac{\cancel{n^2} + \cancel{2n} + 1 - \cancel{n^2} - \cancel{2n}}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$$

$\Rightarrow a_{n+1} > a_n$.

So, $\{\frac{n}{n+1}\}$ is strictly increasing seq.

Ex: $\{(-2)^n\}$ is neither a monolónic increasing or monolonic decreasing seq. This is not a Monolone seq.

Theorem: A ~~do~~ monolonic increasing sequence which is bounded above, is convergent and it converges to the least upper bound (l.u.b)

P.T.O

Proof. Let $\{a_n\}$ be a monotonic increasing seq of real nos which is bounded above. Let M be its least upper bound (l.u.b). Then by defn of l.u.b

i) $a_n \le M \quad \forall n \in \mathbb{N}$

ii) for given $\varepsilon > 0$, $\exists$ $K \in \mathbb{N}$ s.t $a_K > M - \varepsilon$.

Since $\{a_n\}$ is monotonic increasing, so,

$$M - \varepsilon < a_K \le a_{K+1} \le a_{K+2} \le \cdots\cdots \le M$$

$$\Rightarrow M - \varepsilon < a_n \le M < M + \varepsilon \quad \forall n \ge K$$

$$\Rightarrow M - \varepsilon < a_n < M + \varepsilon \ \forall n \ge K.$$

This shows that $\{a_n\}$ is convt and converges to its l.u.b.

Theorem: A monotonic decreasing seq of real number which is bounded below is convergent and converges to its g.l.b.

Proof: left as an excercise.

Problems:

① Show that the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$, $\forall n \ge 1$, converges to 2.

Soln: The sequence is $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\ldots\}$

$$a_{n+1}^2 - a_n^2 = 2a_n - 2a_{n-1} = 2(a_n - a_{n-1})$$

$$\Rightarrow (a_{n+1} + a_n)(a_{n+1} - a_n) = 2(a_n - a_{n-1})$$

Since $a_n > 0$, $a_{n+1} + a_n > 0$, Then $a_{n+1} - a_n \ge 0$ iff $a_n - a_{n-1} \ge 0$

≤ 0 iff ≤ 0.

Again, $a_n - a_{n-1} \ge 0$ or ≤ 0 iff $a_{n-1} - a_{n-2} \ge 0$ or ≤ 0

....

iff $a_2 - a_1 \ge 0$ or ≤ 0.

but $a_2 - a_1 = \sqrt{2\sqrt{2}} - \sqrt{2} > 0$, so. $a_{n+1} > a_n$. So, $\{a_n\}$ is m.i.

Again $a_{n+1}^2 = 2a_n$.

$\Rightarrow 2a_n = a_{n+1}^2 > a_n^2$

$\Rightarrow a_n^2 - 2a_n < 0 \Rightarrow a_n(a_n - 2) < 0 \Rightarrow a_n < 2$ [$\because a_n > 0$].

So, $\{a_n\}$ is bounded above.

So, $\{a_n\}$ is convt as it is bounded above and monotonic increasing.

let $\lim_{n\to\infty} a_n = l$. $\Rightarrow$. $a_{n+1}^2 = 2a_n$

$\Rightarrow \lim_{n\to\infty} a_{n+1}^2 = 2 \lim_{n\to\infty} a_n \Rightarrow l^2 = 2l$.

$\Rightarrow l(l-2) = 0 \Rightarrow l = 2$ [l can not be 0 as $\{a_n\}$ is m.i and $a_1 > 0$] $\rightarrow$ reason is very important.

So., $\{a_n\} \longrightarrow 2$.

(2) Show that the sequence $\{a_n\}$ defined by $a_1 = \sqrt{7}$, and $a_{n+1} = \sqrt{7 + a_n}$ $\forall n \geq 1$ is convergent and find its limit.

Solⁿ: First show, $\{a_n\}$ is monotonic increasing in same way. We now show that $\{a_n\}$ is bounded above,

$a_{n+1}^2 = 7 + a_n$. $\forall n \in \mathbb{N}$.

$\Rightarrow 7 + a_n = a_{n+1}^2 > a_n^2$

or, $a_n^2 - a_n - 7 < 0$

$\Rightarrow (a_n - \alpha)(a_n - \beta) < 0$, where α, β are the roots of $x^2 - x - 7 = 0$. One roof is negative and another roof is positive.

Let $\alpha < 0$.

Since $a_n > 0$ $\forall n$, $a_n - \alpha > 0 \Rightarrow a_n < \beta$ $\forall n \in \mathbb{N}$.

So, $\{a_n\}$ is bounded above by β.

So, $\{a_n\}$ is bounded above + m.i $\Rightarrow$ it is convt.

We now find its limit.

Let $\lim_{n\to\infty} a_n = l$, $a_{n+1}^2 = 7 + a_n$

$$\Rightarrow \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (7+a_n)$$

$$\Rightarrow l^2 = 7+l$$

$$\Rightarrow (l-\alpha)(l-\beta) = 0.$$

But $l \neq \alpha$, Since each element of the sequence is positive and $\alpha < 0$. Therefore $l=\beta$. So, The sequence converges to the positive root of $x^2 - x - 7 = 0$.

Exercise: Use same method to solve

* Prove that the seq $\{a_n\}$ defined by

i) $a_1 = \sqrt{3}$, $a_{n+1} = \sqrt{3a_n}$, $\forall n \geq 1$, converges to 3.

ii) $a_1 = \sqrt{6}$, $a_{n+1} = \sqrt{6+a_n}$ $\forall n \geq 1$, " " to ~~6~~ 3.

** A sequence $\{a_n\}$ is defined by $a_1 > 0$ and $a_{n+1} = \sqrt{6+a_n}$, $\forall n \geq 1$, show that

i) $\{a_n\}$ is monotonic increasing if $0 < a_1 < 3$.

ii) $\{a_n\}$ is monotonic decreasing if $a_1 > 3$.

Problem: Give an example of a sequence of rational numbers that converges to an irrational number.

Solution: $\{(1+\frac{1}{n})^n\}$ is a seq of rational numbers which converges to e. (a irrational number) (See S.K. Mapa for Proof)

P.T.O

Problem 2: Give an example of a sequence of irrational numbe that converges to a rational no.

Sol^n: A sequence $\{a_n\}$ is defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}, \forall n \geq 1$ is a seq of irrational number which converges to 2.

(solved in previous example).

Problem 3: Give an example of divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ is convergent.

Sol^n: $\{a_n\} = \{n\}$, $\{b_n\} = \{-n\}$, $\{a_n + b_n\} = \{0, 0, \ldots\}$ is a constant seq, so it is convt.

Problem 4: Give examples of divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n b_n\}$ is convt.

Sol^n: Exercise

ASSIGNMENT — ASUTOSH COLLEGE

Topic: Sequence of Real Numbers

Choose the correct option with proper justification:

1) Let $\{a_n\}$ be a monotone increasing sequence. Then $\{a_n\}$
   a) always converges
   b) always diverges
   c) converges to its limit sup
   d) oscillates infinitely.

2) Which of the following statement is NOT correct?
   a) Every subsequence of a monotone sequence is monotone
   b) Every monotone sequence has a convergent subsequence.
   c) Every sequence has a monotone subsequence.
   d) Every convergent sequence has a monotone subsequence.

3) Which of the following is NOT a monotone sequence?
   a) $\left\{\frac{n^n}{n!}\right\}$ b) $\left\{\frac{\log n}{n!}\right\}$ c) $\{x_n\}$ where $x_n = \begin{cases} \frac{1}{n^2}, & n \text{ is even} \\ \frac{1}{n}, & n \text{ is odd} \end{cases}$
   d) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

4) If $\{a_n^2\}$ is convergent then $\{a_n\}$
   a) must converge
   b) must diverge
   c) may converge
   d) may diverge to $+\infty$.

5) $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$ equals to

a) e b) $\frac{1}{e}$ c) 1 d) ∞

6) Let $\lim_{n\to\infty} a_n = l_1$, $\lim_{n\to\infty} b_n = l_2$ and $l_1 < l_2$ then

a) $a_n < b_n \;\forall n$

b) $a_n < b_n$ except for finite n.

c) $a_n \geq b_n \;\forall n$

d) $a_n < b_n$ for finite n.

7) If $\{a_n\}$ be a Sequence such that $\left|\frac{a_{n+1}}{a_n} - \frac{1}{2}\right| \to 0$ as $n\to\infty$, then $\lim_{n\to\infty} a_n$ is

a) $\frac{1}{2}$

b) 1

c) 0

d) ∞.

8) Let $\{a_n\}$ be a Sequence such that $a_n \to a$. Let ~~Are~~ $A_K = \left\{n \in \mathbb{N} : |a_n - a| < \frac{1}{K}\right\}$. Then

a) A_K is finite for all $K \in \mathbb{N}$.

b) A_K is finite for all but finitely many K.

c) Each A_K contains all but finitely many Positive integers.

d) $A_K = \mathbb{N} \;\forall K \in \mathbb{N}$

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9) The set of real values of x for which $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ is

a) $\{0\}$ c) $(-1,0)$

b) $(-1,1)$ d) $(-\infty,\infty)$

10) The sequence $\{a_n\}$ of reals, where $0 < a_n < 1$ and $a_n(1-a_{n+1}) > \frac{1}{4}$ $\forall n \in \mathbb{N}$, Converges to

a) 0 b) $\frac{1}{2}$ c) 1 d) 2

11) Let $\{a_n\}$ be a sequence of reals. Then $\lim_{n\to\infty} a_n$ exist if

a) $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+2}$ c) $\lim_{n\to\infty} a_{3n} = \lim_{n\to\infty} a_{2n}$

b) $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1}$ d) none of these.

12) Let $\{a_n\}$ be a real sequence such that $a_{n+1} = 2 - \frac{1}{a_n}$, $n \geq 1$ where $a_1 > 1$. Then $\{a_n\}$ is

a) bounded but not monotone.

b) not bounded but monotone.

c) both bounded and monotone.

d) neither bounded nor monotone.

Note: The note of Subsequence of real nos will be uploaded in time.

Ghosh