Study Material

Subject: Mathematics

Semester: 4th

Name of Teacher: Prabir Rudra

Topic: Mechanics (Particle Dynamics) (CC-10)

Advice from faculty

These notes carry both theory and problems on **Dynamics of a particle under a force field** particularly **a central force field (CC-10)**. The students are advised to read the relevant chapter from any standard text book they have at their disposal. In case they do not have any text book they can inform via the whatsapp group and I will try to provide an e-book. After reading the chapter from a text book then concentrate on these notes to get a better understanding. In case you have any query regarding the topic you may consult me via e-mail (prudra.math@gmail.com) or whatsapp.

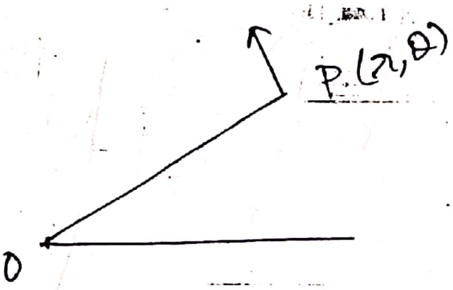
I think this will be sufficient material for 10 days at least. I will be back with the next topic after 10th April, 2020.

Date: 28/03/2020

PARTICLE DYNAMICS.

Central force:

central force → acting towards or away from a point



$\dot{r} = \frac{dr}{dt}$ $\qquad \ddot{r} - r\dot{\theta}^2$

$r\dot{\theta} = r\frac{d\theta}{dt}$ $\qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

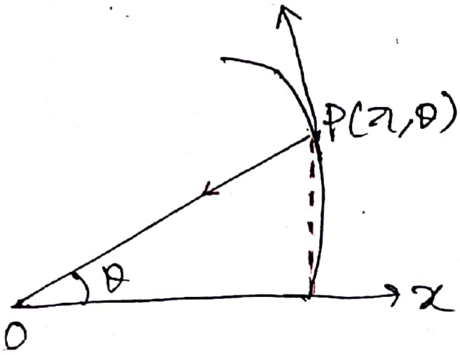
Central force: Let F be a force having the following characteristics

(i) It is always directed towards or away from a fixed point.

(ii) The magnitude of the force is a function of the distance 'r' along.

Such a force is called a central force and the fixed point is called the centre of the force. The path described by the particle is called the central orbit.

The central orbit is a plane curve.



With the centre of force as pole and a fixed line $\overrightarrow{OX}$ through O as the initial line let P(r,θ) be the polar coordinates of the moving particle at any time 't'. If F(r) be the central force per unit mass then. The eqn. of motion is Cartesian coordinates is

$$m\frac{d^2x}{dt^2} = -mF(r)\frac{x}{r} \quad \text{——} \quad (1)$$

$$m\frac{d^2y}{dt^2} = -mF(r)\frac{y}{r} \quad \text{——} \quad (2)$$

$$m\frac{d^2z}{dt^2} = -mF(r)\frac{z}{r} \quad \text{——} \quad (3)$$

By (2) × z − (3) × y we get

$$y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} = 0$$

or, $\frac{d}{dt}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = 0$

or, $y\frac{dz}{dt} - z\frac{dy}{dt} = A$ (say) —— (4)

And $z\frac{dx}{dt} - x\frac{dz}{dt} = B$ —— (5)

And $x\frac{dy}{dt} - y\frac{dx}{dt} = C$ —— (6)

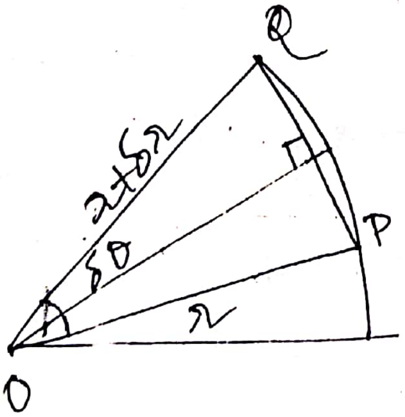
by (4) $\times x$ + (5) $\times y$ + (6) $\times z = 0$

or, $Ax + By + Cz = 0$

Thus the curve described by the particle satisfies the eqn. $Ax + By + Cz = 0$ which is a plane.

N.B: $F(r)$ = central force per unit mass i.e., called central acceleration.

Areal velocity



ΔOPQ

When a particle describes a plane curve under the action of a central force the rate of change of the area traced out by the radius vector is called the Areal velocity of the particle.

$$\text{Areal velocity} = \lim_{\Delta t \to 0} \frac{\Delta OPQ}{\Delta t}$$

$$= \frac{1}{2} \lim_{\delta t \to 0} \frac{r(r+\delta r)\sin\delta\theta}{\Delta t} = \frac{1}{2} \lim_{\Delta t \to 0} r(r+\Delta r)\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta t}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{constant (let)} \quad \text{—(1)}$$

Again if p be the length of the perpendicular from O on the st. line PQ and v be the velocity of the particle at P then the areal velocity is

$$\lim_{\delta t \to 0} \frac{1}{2} p \frac{ds}{dt} = \frac{1}{2} pv \quad \text{—— (2)}$$

So $r^2\dot{\theta} = pv = \text{constant}$.

(III) Angular momentum & kinetic energy:

In polar form $r\dot{\theta}$ is the crossradial velocity. ~~Oom~~ So.

moment of momentum about O is $mr^2\dot{\theta}$.

kinetic energy $= \frac{1}{2}mv^2$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Differential eqn. of the path in polar coordinates:

The eqn. of motion of the particle whose position at any time is given by the coordinates $(r, \theta) \rightarrow$

$$m(\ddot{r} - r\dot{\theta}^2) = -mf \quad \text{---} (1)$$

where F is central force per unit mass towards the centre

and $m \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) = 0$ ——— (2)

From (2) $r^2\dot{\theta} = h$ (say)

$$\dot{\theta} = \frac{h}{r^2} = hu^2 \quad \text{where } u = \frac{1}{r}$$

So $\dot{r} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\cdot\dot{\theta}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

Putting these values of $r, \ddot{r}$ and $\dot{\theta}$ in ① we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -F$$

or, $$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}} \quad \text{--- ③}$$

This is the differential eqn. of the path.

(ii) Differential eqn. of the path in pedal coordinates:

Let p be the perpendicular distance of the tangent at P to the path from O. If coordinates of P (r, θ)

$$1/p^2 = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

As $r = \frac{1}{u}$, $\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

Differentiating w.r.t r we get

$$-\frac{2}{p^3}\frac{dp}{dr} = 2u\frac{du}{dr} + 2\frac{du}{d\theta}\frac{d}{dr}\left(\frac{du}{d\theta}\right)$$

$$-\frac{1}{p^3}\frac{dp}{dr} = u\cdot\frac{du}{dr} + \frac{du}{d\theta}\cdot\frac{d^2u}{dr\,d\theta}$$

$$= \frac{du}{dr}\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$\left[\begin{array}{l} u = \frac{1}{r} \\ \frac{du}{dr} = -\frac{1}{r^2} \end{array}\right]$$

$$-\frac{1}{p^3}\frac{dp}{dr} = \frac{du}{dr}\frac{F}{h^2u^2} = -\frac{F}{r^2u^2h^2} = -\frac{F}{h^2}$$

$$\boxed{\frac{h^2}{p^3}\frac{dp}{dr} = F} \quad \text{--- (4)}$$

This is the diff eqn. of the path in pedal coordinates

$$vp = h \qquad \dot{\theta} = \frac{h}{r^2}$$

$$v = \frac{h}{p}$$

N.B: In a central orbit the angular velocity varies as the square of the radius and the linear velocity varies inversely as the perpendicular distance of the tangent at that point.

Law of forces for an elliptical orbit :

The eqn. of an ellipse with focus as pole is $\frac{l}{r} = 1 + e\cos\theta$

$e < 1$

$$u = \frac{1}{l} + \frac{e}{l}\cos\theta$$

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta$$

$$\frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$u + \frac{d^2u}{d\theta^2} = \frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta$$

$$= \frac{1}{l}$$

$$\therefore F = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$= \frac{h^2u^2}{l} = \frac{h^2}{l}\cdot\frac{1}{r^2} = \frac{\mu}{r^2}$$

Where $\mu = \frac{h^2}{l}$

And $vp = h$

ith

cos θ

So $v^2 = \frac{h^2}{p^2} = h^2\left\{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}$

$= h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}$ $\left[\because -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta}\right]$

$= \frac{h^2}{L^2}\left\{(1+e\cos\theta)^2 + e^2\sin^2\theta\right\}$

$= \frac{h^2}{L^2}\left\{1+e^2+2e\cos\theta\right\}$

$= \mu\left\{2.\frac{1+e\cos\theta}{L} - \frac{1-e^2}{L}\right\}$

$= \boxed{\mu\left\{\frac{2}{r} - \frac{1}{a}\right\}}$

cos θ

This shows that the velocity of the particle at any point depends on 'r' only and independent of direction of motion

If T be the time taken to describe the whole area of the ellipse

$h = r^2\dot{\theta}$

$\frac{h}{2} \Rightarrow$ rate of description of sectorial area.

$T = \frac{h}{2}.T = \pi ab$

or, $T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu L}} = \frac{2\pi ab}{\sqrt{\mu\frac{b^2}{a}}}$

$\frac{2\pi}{\sqrt{\mu}} a^{3/2}$

For a parabolic orbit with centre of force at the focus

$F = \frac{\mu}{r^2}$ $\qquad v^2 = \frac{2\mu}{r}$

$F = \frac{\mu}{r^2}$ $\qquad v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$

A particle describes the parabola $p^2 = ar$ under a force which is always directed towards its focus. (1)

Find the Law of force.

$$\frac{h^2}{p^3}\frac{dp}{dr} = F$$

And

$$p^2 = ar$$

$$2p\frac{dp}{dr} = a$$

$$\frac{h^2}{p^3}\frac{a}{2p} = F$$

$$F = \frac{ah^2}{2p^4} = \frac{ah^2}{2a^2r^2} = \frac{h^2}{2ar^2} = \frac{h^2}{2a} \cdot \frac{1}{r^2}$$

centre

$$vp = h \Rightarrow v^2p^2 = h^2$$

$$v^2 = \frac{h^2}{p^2} = \frac{h^2}{ar}$$

$$v = \frac{h}{\sqrt{ar}}$$

$\frac{1}{a})$

(1) Find the law of force for the path $r^4 = a^4 \cos 4\theta$ and the velocity at any point.

Soln. Taking Log of the eqn.

We get

$$4 \log r = 4 \log a + \log \cos 4\theta$$

diffn. w.r.t θ we get $\Rightarrow$

$$\Rightarrow \frac{4}{r}\frac{dr}{d\theta} = -\frac{4\sin 4\theta}{\cos 4\theta} = -4\tan 4\theta$$

$$-\frac{1}{r}\frac{dr}{d\theta} = \tan 4\theta$$

or, $\cot\phi = -\tan 4\theta$

or, $\tan\phi = -\cot 4\theta$

$$= \tan\left(\frac{\pi}{2} + 4\theta\right)$$

$$\Rightarrow \phi = \frac{\pi}{2} + 4\theta$$

And $p = r\sin\phi$

$= r\cos 4\theta = \frac{r^{5}}{a^4}$

And $\frac{dp}{dr} = \frac{5r^4}{a^4}$ $(n+1)\frac{r^n}{a^n}$

$\frac{h^2}{p^3}\frac{dp}{dr} = F$

$\Rightarrow \frac{h^2}{r^{15}} a^{12} \cdot \frac{5r^4}{a^4} = F$

And $5a^8 h^2 r^{-11} = F$

or, $F \propto r^{-11}$

$pv = h$

or, $v = \frac{h}{p} = \frac{ha^4}{r^5}$

And $v \propto r^{-5}$

(2) (III) A particle describes a central orbit $r^n = a^n \cos n\theta$ under a force directed towards the pole. Find the law of force and velocity at any position

Sol^n:

Now $F = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2}\right)$

$$= \frac{h^2 (n+1)\, a^{2n}}{r^{2n+3}}$$

And $v^2 = h^2 \left\{ u^2 + \left(\frac{du}{d\theta}\right)^2 \right\}$

$$= \frac{h^2 a^{2n}}{r^{2n+2}}$$

Special cases:

$\boxed{\text{if } n = -\frac{1}{2}}$

$r^{-1/2} = a^{-1/2} \cos\left(\frac{\theta}{2}\right)$ is the eq^n.

So $r^{-1/2} = a^{-1/2} \cos\frac{\theta}{2}$

And squaring both sides

$$r^{-1} = a^{-1} \cos^2\frac{\theta}{2}$$

And $r = \frac{a}{\cos^2\frac{\theta}{2}}$

$$r = \frac{2a}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{2a}{(\cos\theta + 1)}$$

And $\frac{2a}{r} = 1 + \cos\theta$

Comparing with $\frac{L}{r} = 1 + e\cos\theta$

$e=1$ So the eqn. is of a parabola (focus as pole)

$\therefore \; F \propto \frac{1}{r^2}$

$v^2 \propto \frac{1}{r}$

$\boxed{\text{if } n=\frac{1}{2}}$

$r^{1/2} = a^{1/2} \cos\frac{\theta}{2}$

Squaring both sides and multiplying by 2 we get

$2r = 2a\cos^2\frac{\theta}{2}$

$= a(1+\cos\theta)$

And $r = \frac{a}{2}(1+\cos\theta)$

This is eqn. of cardiode.

$F \propto \frac{1}{r^4}$

~~And so~~

$\boxed{\text{If } \; n=1}$

$r = a\cos\theta$

$\Rightarrow \; F \propto \frac{1}{r^5}$

$\boxed{\text{If } n=2}$

The eqn. is $r^2 = a^2 \cos 2\theta$ catenary

$$F \propto \frac{1}{r^7}$$

$\boxed{\text{if } n=-2}$

$$r^{-2} = a^{-2} \cos(-2\theta)$$

$$r^2 = \frac{a^2}{\cos 2\theta}$$

or, $$r^2 \cos 2\theta = a^2$$

Rectangular hyperbola

$$F \propto r$$

If the central orbit is an ellipse the focus being the centre of force. The prove that the time average of the reciprocal distance is the reciprocal of the semimajor length a. Deduce further that the time average of v^2 is $\frac{\mu}{a}$ i.e.

(i) $\frac{1}{T}\int \frac{dT}{r} = \frac{1}{a}$

(ii) $\frac{1}{T}\int v^2 dT = \frac{\mu}{a}$

In both the cases the integration are for a complete integration

$$I = \oint \frac{dt}{r}$$

$$= \oint \frac{1}{r}\cdot \frac{d\theta}{\frac{d\theta}{dt}}$$

$$= \oint \frac{1}{r}\frac{d\theta}{\dot\theta} = \oint \frac{r\, d\theta}{r^2\dot\theta}$$

$$= \frac{1}{h}\oint r d\theta$$

$$= \frac{l}{h}\int_0^{2\pi} \frac{d\theta}{1+e\cos\theta}$$

$$\left[\begin{array}{l} \text{As } \frac{l}{r} = 1+e\cos\theta \\ \text{and } r = \frac{l}{1+e\cos\theta} \end{array}\right]$$

$$= \frac{l}{h}\int_0^{2\pi} \frac{d\theta}{1+e\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)}$$

$$= \frac{l}{h}\int_0^{2\pi} \frac{d\theta}{(1-e)\sin^2\frac{\theta}{2} + (1+e)\cos^2\frac{\theta}{2}}$$

$$= 2\cdot\frac{l}{h}\int_0^{\pi} \frac{\sec^2\frac{\theta}{2}\, d\theta}{(1+e) + (1-e)\tan^2\frac{\theta}{2}}$$

Let $\tan\frac{\theta}{2} = z$

Then $\frac{1}{2}\sec^2\frac{\theta}{2} = \frac{dz}{d\theta}$

And

θ	π	0
z	∞	0

So $$I = \frac{2l}{h}\int_0^{\infty} \frac{2\, dz}{(1+e)+(1-e)z^2}$$

$$= \frac{4l}{h}\int_0^{\infty} \frac{dz}{(1+e)+(1-e)z^2}$$

$$= \frac{4l}{h(1-e)} \int_0^\infty \frac{dz}{\left(\sqrt{\frac{1+e}{1-e}}\right)^2 + z^2}$$

$$= \frac{4l}{h(1-e)} \sqrt{\frac{1-e}{1+e}} \left[\tan^{-1} \frac{z}{\sqrt{\frac{1+e}{1-e}}} \right]_0^\infty$$

$$= \frac{4l}{h\sqrt{1-e^2}} \frac{\pi}{2}$$

$$= \frac{2l\pi}{h\sqrt{1-e^2}}$$

$$= \frac{2l\pi}{h} \cdot \frac{a}{b}$$

$$= \frac{2b^2\pi}{ah} \cdot \frac{a}{b} = \frac{2b\pi}{h}$$

So $\displaystyle \frac{1}{T}\int_0^T \frac{dt}{r} = \frac{2\pi b}{h} \cdot \frac{h}{2\pi ab} = \frac{1}{a}$

$\int_0^\infty$

And $\frac{1}{T}\int_0^T v^2 dt$

$$= \frac{\mu}{T}\int_0^T \left(\frac{2}{r} - \frac{1}{a}\right) dt$$

$$= \frac{2\mu}{a} - \frac{\mu}{a} = \frac{\mu}{a}$$

$= \frac{1}{a}$

(11) If the central orbit be an ellipse under the force towards the centre. Find the law of force and the velocity at any time.

Solⁿ

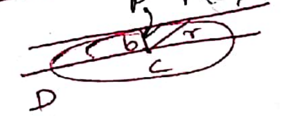
$$\frac{a^2b^2}{p^2} = a^2+b^2-r^2$$

Differentiating both sides with respect to r we get

$$-\frac{2a^2b^2}{p^3}\frac{dp}{dr} = -2r$$

$$\Rightarrow \frac{a^2b^2}{p^3}\frac{dp}{dr} = r$$

And $\frac{dp}{dr} = \frac{rp^3}{a^2b^2}$



$CD^2 + CD^2 = a^2+b^2$

$p \times CD = ab$

$\frac{a^2b^2}{p^2} = a^2+b^2-r^2$

CD the semidiameter conjugate to CP.

Again we know

$$F = \frac{h^2}{p^3}\frac{dp}{dr}$$

$$= \frac{h^2}{p^3}\frac{p^3}{a^2b^2}\cdot r$$

$$= \frac{h^2}{a^2b^2}\cdot r$$

so $F \propto r$

And $pv = h$

$\Rightarrow$ ~~And~~ $v^2 = \frac{h^2}{p^2}$

$$= \frac{h^2(a^2+b^2-r^2)}{a^2b^2}$$

(11) In an orbit described under a force to a centre the velocity at any pt is inversely proportional to the distance of the point from the centre of the force: Show that the path is an equiangular spiral.

Sol^n

$$v = \frac{A}{r}$$

Here $v \propto \frac{1}{r} \Rightarrow h = pv \Rightarrow p = \frac{h}{v} \Rightarrow p^2 = \frac{h^2 r^2}{A^2} \Rightarrow \frac{1}{p^2} = \frac{\mu}{h^2 r^2} = Ku^2$

$\because \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \Rightarrow$

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{\mu}{h^2 r^2} = Ku^2$$

or, $$\frac{du}{d\theta} = \sqrt{(K-1)}\, u$$

$$\Rightarrow \frac{du}{u} = \sqrt{(K-1)}\, d\theta$$

$$\Rightarrow \ln u = \sqrt{K-1}\,\theta + a$$

a = constant

or $$u = Ae^{\sqrt{K-1}\,\theta}$$

A is another arbitrary constant.

So $$r = \frac{1}{A} e^{-\sqrt{(K-1)}\,\theta}$$

The equiangular spiral.

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(11) Show that the only law of acc^n which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to that distance is that of inverse case.

Sol^n. For a circular orbit

$v^2 = Fr$ [from centripetal acc^n]

or, $-2\int_{\infty}^{r} F\,dr = Fr$ [Force towards the centre]

$-2F = F + r\frac{dF}{dr}$

or, $-3F = r\frac{dF}{dr}$

or, $\frac{dF}{F} = -3\frac{dr}{r}$

Solving

$\ln F = -3\ln r + \ln c$

$F = \frac{c}{r^3}$ $\quad c = $ Arb const.

$\Rightarrow F \propto \frac{1}{r^3}$

(ii) Velocity in circle:

When a particle describes a circle of radius r about a centre of force of attraction then the velocity v at any point in the circle is given by the relation $\frac{v^2}{r} = F$

(Accn. towards the centre)

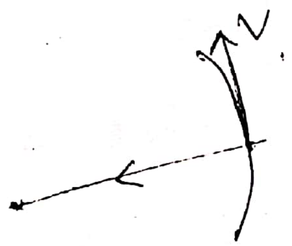
This velocity is often referred to as the velocity in a circle.

(iii) Velocity from infinity:

If a particle falls from rest at infinity to a point under the action of a given attractive force $F(r)$ associated with the orbit then the velocity acquired by the particle is referred to as velocity from infinity at that point.

If v be the velocity at any point on the orbit then describing along the tangent we get

$$-F(r)\frac{dr}{ds} = v\frac{dv}{ds}$$



$$\int_0^V v\,dv = -\int_\infty^r F(r)\,dr$$

or, $$\frac{V^2}{2} = -\int_\infty^r F(r)\,dr$$

or, $$\frac{1}{2}V^2 = -\int_\infty^r \mu\frac{dr}{r^2}$$

$$= \left[\frac{\mu}{r}\right]_\infty^r$$

$$= \frac{\mu}{r}$$

or, $$\boxed{V = \sqrt{\frac{2\mu}{r}}}$$

(iii) Velocity to the origine:

If a particle moves from rest at a given position to the centre of force under the given central force of accn $F(r)$ then the velocity acquired by the particle is called the velocity to the origine at that point.

Thus if v be the velocity to the origine at a distance 'a' ~~fro~~ from the centre,

then,

$$\frac{1}{2}v^2 = -\int_a^0 F(r)\,dr$$

$$\Rightarrow \quad v^2 = -2\int_a^0 F(r)\,dr = 2\int_0^a F(r)\,dr$$

~~Esp of~~

Especially if $F = \mu r$

Then $v^2 = 2\mu\int_0^a r\,dr = \mu a^2$.