Study Material

Subject: Mathematics

Semester: 4th

Name of Teacher: Prabir Rudra

Topic: Mechanics (Particle Dynamics-part4) (CC-10)

Advice from faculty

These notes carry both theory and problems on **Oscillations and Simple Harmonic Motion**. The students are advised to study these notes to get an initial understanding of the topic. In case you have any query regarding the topic you may consult me via e-mail (prudra.math@gmail.com) or WhatsApp. **We are soon going to arrange an online doubt clearing session via video conference on the topic. The date and time will be informed soon.**

I think this will be sufficient material for 10 days at least. I will be back with the next topic after 20th May, 2020.

Date: 09/05/2020

Chapter Three

Simple Harmonic Motion

3.1. Introduction

In Chapter two, we have discussed the motion in a straight line under variable acceleration and different initial conditions of motion. In this chapter we shall discuss a special type of motion in a straight line, what is known as Simple Harmonic Motion. Before defining simple harmonic motion, let us explain the periodic motion and oscillatory motion. When a body repeats its motion continuously on a definite path in a definite interval of time, its motion is called a 'periodic motion' and the interval of time is called time-period. For example, moon completes one revolution around the earth in 27.3 days. The motion of the moon is periodic with time period 27.3 days. The motions of the hands of a clock are also periodic. The time-period of the minute hand is one hour and that of the hour hand is 12 hours. If a body in periodic motion moves along the same path to and fro about a fixed point then the motion of the body is an oscillatory motion (or vibratory motion), the fixed point is called the mean position or equilibrium position. The motion of the pendulum of a wall clock, the motion of the bob of a simple pendulum, the motion of a bar magnet suspended in earth's magnetic field are some examples of oscillatory motion. From above examples it follows that all oscillatory motions are periodic motions, but all periodic motions are not oscillatory. The revolution of Earth round the Sun is a periodic motion but not an oscillatory motion. The simplest type of oscillatory motion is simple harmonic motion.

Definition (Simple Harmonic Motion) : When a particle moves in a straight line with an acceleration which is always directed towards a fixed point on the line and is proportional to the distance of the particle from the fixed point then the motion of the particle will be oscillatory motion and this motion is known as simple harmonic motion or briefly S.H.M.

3.2. Simple Harmonic Motion and its solution.

Suppose a particle starts from rest at a distance a from a fixed point O on a straight line and moves along the line X'OX under acceleration which is always directed towards O and varies as the distance of the particle from O. We shall investigate the motion analytically.

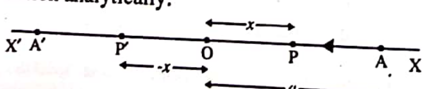

Fig. 3.1

Let at time t, P be the position of the particle of mass m and its initial position was at A at a distance $|OA| = a$ when its velocity was zero. Here time t is measured from the instant when it is at A.

Let $|OP| = x$. Then the acceleration of the particle at P is $m\,\mu\,x$ directed towards O, where $\mu\ (> 0)$ is a constant (fig. 3.1).

Then equation of motion of the particle is $m\dfrac{d^2x}{dt^2} = -m\,\mu\,x$ ... (3.2.1)

negative sign is taken on the right hand side, since the acceleration is positive in the direction of x increasing while in this case x is decreasing in the direction of acceleration.

Multiplying both sides of (3.2.1) by $2\dfrac{dx}{dt}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = c_1 - \mu x^2 \qquad \text{... (3.2.2)}$$

c_1 is a constant of integration.

Using initial condition of motion : $x = a$, $\dfrac{dx}{dt} = 0$ when $t = 0$, we have $c_1 = \mu a^2$.

Then $\left(\dfrac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$

or, $$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2 - x^2} \quad (0 \le x \le a) \qquad \text{... (3.2.3)}$$

negative sign is taken on the right hand side of (3.2.2), since velocity is positive in the direction of x increasing, while in this case x is decreasing.

From (3.2.3), $$dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2 - x^2}}.$$

Integrating, $$t = c_2 + \frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{x}{a}\right) \qquad \text{... (3.2.4)}$$

At $t = 0$, $x = a$, hence $c_2 = 0$.

Thus from (3.2.4), we have $x = a\cos\sqrt{\mu}\,t$ ... (3.2.5)

Relations (3.2.3) and (3.2.5) respectively give the velocity and distance covered at time t for the motion from A to O

If t_1 be the time from A to O then, from (3.2.5), $0 = a\cos\sqrt{\mu}\,t_1 \Rightarrow t_1 = \dfrac{\pi}{2\sqrt{\mu}}$.

In the motion from A to O, its velocity will be maximum (numerically) on reaching O, given by $V = -a\sqrt{\mu}$, which can be obtained by taking $x = 0$ in (3.2.2).

Since $V < 0$, this implies the particle will cross O and move on the left side of O.

As the acceleration is always directed towards O, its velocity will begin to diminish (numerically).

If P′ be the position of the particle on the left of O at any time t such that $|OP'| = |OP| = x\ (> 0)$ then $OP' = -x < 0$ and the acceration of the particle is $\mu(-x)$ directed towards O. Since the acceleration is directed towards O and the particle is moving away from O on the negative side of O, so equation of motion of the particles is now

$$m\frac{d^2(-x)}{dt^2} = -m\mu(-x) \quad \text{or,} \quad \frac{d^2x}{dt^2} = -\mu x, \text{ which is same as equation (3.2.1).}$$

Thus whether the particle is on the right side, or left side of O, we get the same equation of motion and hence the velocity will be destroyed at the same rate at which it increased during the motion from A to O.

Hence velocity of the particle will be zero again when $x = -a$, which is obtained from (3.2.2). So the particle will have zero velocity at A′, where $|OA'| = |OA| = a$.

At A′, the acceleration is maximum and is directed towards O. So exactly same motion will be repeated in a reverse direction and the particle again comes to rest at A, when the acceleration is directed towards O. So this motion will be repeated over and over again. Such a motion of the particle is oscillatory about the mean position O.

This oscillatory motion is called **Simple Harmonic Motion (S.H.M).**

The maximum distance of the particle on either side of O i.e. 'a' is called amplitude of the oscillation and O is called centre of oscillation.

The total time from A to A′ and back to A is four times the time from A to O, since the motion is symmetrical about O, and is given by $T = 4t_1 = \frac{2\pi}{\sqrt{\mu}}$. T is called the time-period or simply the period of oscillation. Note that T is independent of the amplitude of oscillation.

General solution of S.H.M.

The equation (3.2.1) can be written as $(D^2 + \mu)x = 0$ ... (3.2.6), where $D \equiv \frac{d}{dt}$.

General solution of the equation (3.2.6) is $x = c' \cos\sqrt{\mu}\,t + c'' \sin\sqrt{\mu}\,t$ where c', c'' are arbitrary constants. Taking $c' = -c\cos\varepsilon$, $c'' = c\sin\varepsilon$, we have

$$x = c\cos(\sqrt{\mu}\,t + \varepsilon) \qquad \text{... (3.2.7)}$$

where c and ε are arbitrary constants.

From (3.2.7) $\qquad \frac{dx}{dt} = -c\sqrt{\mu}\,\sin(\sqrt{\mu}\,t + \varepsilon) \qquad$ ... (3.2.8)

We now impose the condition that at time $t = t_0$, the particle was at A and it started from rest. Thus time t is measured not from the instant when the particle was at A but it is measured from some other instant.

Then from (3.2.7) and (3.2.8),

$a = c\cos(\sqrt{\mu}\ t_0 + \varepsilon)$ and $\sin(\sqrt{\mu}\ t_0 + \varepsilon) = 0$.

$\Rightarrow \sqrt{\mu}\,t_0 + \varepsilon = 0$ i.e. $t_0 = -\frac{\varepsilon}{\sqrt{\mu}}$. Then $c = a$.

Thus general solution of the equation (3.2.6) is

$$x = a\cos(\sqrt{\mu}\,t + \varepsilon) \qquad \text{... (3.2.9)}$$

and hence $\qquad \frac{dx}{dt} = -a\sqrt{\mu}\,\sin(\sqrt{\mu}\,t + \varepsilon) \qquad$ ... (3.2.10)

As t varies, we have from (3.2.9), $|x| \le a$, since $\cos(\sqrt{\mu}\,t + \varepsilon)$ changes periodically between -1 and $+1$. Then $-a \le x \le a$, which implies that the motion of the particle is oscillatory, oscillating about O. This oscillatory motion is known as simple harmonic motion.

The greatest distance a, the particle moves on either side of the centre of oscillation O, is known as **amplitude** of the motion.

From (3.2.9) and (3.2.10) we note that both $\cos(\sqrt{\mu}\,t + \varepsilon)$ and $\sin(\sqrt{\mu}\,t + \varepsilon)$ are periodic functions of period $\frac{2\pi}{\sqrt{\mu}}$. Hence after every period of $\frac{2\pi}{\sqrt{\mu}}$, the particle passes through the same position with same velocity in the same sense. Thus the same motion is repeated over and over again at intervals of $\frac{2\pi}{\sqrt{\mu}}$. This time interval $\frac{2\pi}{\sqrt{\mu}}$ is called the **time period** of oscillation.

The number of complete oscillations per unit time is $n = \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$ and n is called **frequency** of oscillation.

After one complete oscillation about O (starting from A and back to A) the angular distance covered around O is twice linear angle i.e. 2π. Hence the angular distance covered per unit time is $\frac{2\pi}{T} = \sqrt{\mu}$. $\sqrt{\mu}$ is called **angular frequency** of the oscillation, which gives interpretation of μ in S.H.M.

The quantity ε is called the **phase angle** or **epoch** and $\sqrt{\mu}\, t + \varepsilon$ is called the **argument**.

From (3.2.9), since at $t = t_0$, $x = a$ we have $t_0 + \dfrac{\varepsilon}{\sqrt{\mu}} = 0$.

The amount of time that has elapsed, since the particle was at its extreme position A when $x = a$ and $t = t_0$, is $t - t_0 = t + \dfrac{\varepsilon}{\sqrt{\mu}} = \dfrac{\sqrt{\mu}t + \varepsilon}{\sqrt{\mu}}$, which is called the **phase** of the motion at time t.

Remark : General solution of the equation (3.2.6) can also be written as $x = a \sin(\sqrt{\mu}\, t + \varepsilon)$, by taking $c' = a \sin\varepsilon$, $c'' = a \cos\varepsilon$ in $x = c' \cos\sqrt{\mu}\, t + c'' \sin\sqrt{\mu}\, t$.

3.3. Geometrical Representation of S.H.M.

Suppose a particle moves in a circle of radius a and centre at O with uniform angular velocity ω. Suppose it starts from the point A at time $t = 0$ from rest and at time t, P be the position of the particle on the circle.

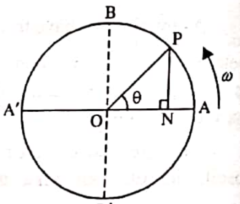

Fig. 3.2

Let $\angle AOP = \theta$ and N be the foot of the perpendicular on the diameter AA'.

Then $x = ON = a \cos\theta = a \cos \omega t$, since $\dfrac{d\theta}{dt} = \omega$ and $\theta = 0$ when $t = 0$,

and $\dfrac{dx}{dt} = -a\omega \sin \omega t = -\omega\sqrt{a^2 - x^2}$, $\dfrac{d^2x}{dt^2} = -a\omega^2 \cos \omega t = -\omega^2 x$.

As the particle moves ahead of P, when it reaches the point B, its projection on the diameter is O, where $x = 0$.

Thus when the particle moves from A to B along the circle, the point N moves along the diameter from A to O.

Similarly, when the particle moves from B to A', the point N moves from O to A'. When it moves from A' to A, the point N will begin to retrace the path from A' to A. So for one complete rotation of the particle along the circle with uniform angular velocity, the point N executes oscillatory motion along the straight line AA' about the centre of oscillation O (See fig. 3.2).

Thus the motion of N along the diameter is S.H.M. Since for one complete rotation about O, angular distnace covered is 2π in time T (say) i.e. $\omega T = 2\pi$ then $T = \dfrac{2\pi}{\omega}$, the time period of S.H.M.

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So, if a particle moves with uniform angular velocity along the circumference of a circle then the straight-line motion of the projection of the particle on a diameter of the circle is called Simple Harmonic Motion. The circle is called the reference circle of the S.H.M.

3.4. Oscillation of a particle attached to an elastic string in horizontal and vertical positions.

When an elastic string is attached to a fixed support at one end and a particle attached to its other end is pulled, then the string is stretched.

If 'l' be the natural (i.e., unstretched) length of the string and x be the stretched length of the the string then extension of the string is $x - l$. When the tied mass is released from the pull, a restoring force T will act to play on the mass, which is directed towards the fixed end of the string. Hooke's law states that, this restoring force T directly varies with the per unit extension of the string. Mathematically $T \propto \dfrac{x-l}{l}$ or, $T = \lambda \dfrac{x-l}{l}$; constant of proportionality λ is called modulus of elasticity of the string. T is called the tension of the string.

When $x = 2l$, then $T = \lambda$. Thus λ for a string of unit cross-section is equal to the amount of force which would stretch it to twice its natural length.

(i) Oscillation in horizontal position of the string.

Let OA = a be the natural length of an elastic string which is placed on a smooth horizontal table.

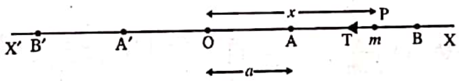

Fig. 3.3

Let a particle of mass m be tied to the end A of the string and the end O is fixed on the table. Suppose the particle is pulled upto a point B on the table such that OB = b and is then let go. Let at time t measured from B where $t = 0$, P be the position of the particle such that OP = x. Then extension of the string is AP = $x - a$. The force acting on the particle at P is tension $T = \lambda \dfrac{x-a}{a}$ directed towards O.

Equation of motion of the particle is

$$m \frac{d^2x}{dt^2} = -T \text{ (negative sign, since T acts in the direction of } x \text{ decreasing)}$$

or,
$$\frac{d^2x}{dt^2} = -\frac{\lambda}{am}(x - a) \qquad \text{... (3.4.1)}$$

Let $y = x - a$. Then equation (3.4.1) reduces to

$$\frac{d^2y}{dt^2} = -\frac{\lambda}{am}y \qquad \text{... (3.4.2)}$$

which shows that the particle executes S.H.M. with $y = 0$ i.e. $x = a$(i.e. A) is the centre of oscillation.

The general solution of (3.4.2) is

$$y = c\cos\left(\sqrt{\frac{\lambda}{am}}t + \varepsilon\right) \qquad \text{... (3.4.3)}$$

where c, ε are arbitrary constants.

Since B is the position of instantaneous rest, so when $t = 0$, $x = b$, $\frac{dx}{dt} = 0$.

Then at $t = 0$, $y = b - a$, $\frac{dy}{dt} = 0$. So, from (3.4.3) we have $b - a = c\cos\varepsilon$ and $\sin\varepsilon = 0$.

Thus $\varepsilon = 0$ and $c = b - a$.

Hence from (3.4.3), we have

$$y = (b-a)\cos\sqrt{\frac{\lambda}{am}}\cdot t$$

or,
$$x = a + (b-a)\cos\left(\sqrt{\frac{\lambda}{am}}\cdot t\right) \qquad \text{... (3.4.4)}$$

and
$$\frac{dx}{dt} = -(b-a)\sqrt{\frac{\lambda}{am}}\sin\left(\sqrt{\frac{\lambda}{am}}\cdot t\right) \qquad \text{... (3.4.5)}$$

If t_1 be the time to reach the point A from B then $x = a$ at $t = t_1$. So, from (3.4.4), we have $t_1 = \frac{\pi}{2}\sqrt{\frac{am}{\lambda}}$. Then from (3.4.5), the velocity of the particle on reaching A is $v_1 = \left(\frac{dx}{dt}\right)_{x=a} = -(b-a)\sqrt{\frac{\lambda}{am}}$.

The fact $v_1 < 0$ shows that the particle will cross the point A on its left side and as the string becomes slack, tension ceases to act (see fig. 3.3).

Thus the motion of the particle on the left of A is an uniform motion with constant velocity v_1 till it reaches a point A′ such that OA = OA′ = a, when the string regains its natural length and velocity at A′ is v_1 directed away from O. So the particle will cross the point A′ on its left and will move away from A′ under tension which is directed towards O. This tension will destroy the velocity v_1 at A′ and will

compel the particle to be instateneously at rest at B′ again where OB′ = OB, since the equation of motion of the particle will be same as given by (3.4.1).

From symmetry, time from A′ to B′ is same as the time from B to A. On reaching B′, tension is maximum and directed towards O, while velocity is zero. So the particle will retrace its path from B′ to A′ and will reach A′ with velocity $v_2 = (b-a)\sqrt{\frac{\lambda}{am}}$ directed towards O, which is numerically equal to v_1. Tension at A′ ceases to act and the particle will move with uniform velocity v_2 till it reaches the point A, when the string regains its natural length. As $v_1 = v_2$ (numerically), time from A′ to A is same as time from A to A′.

At A, the particle will move with velocity v_2 against the tension acting towards O and the motion will be guided by the same equation (3.4.1). So, finally the particle will be at instantaneous rest at B again, the tension is at its maximum value directed towards O and hence the motion will be repeated over and over again.

Thus we see that the motion is oscillatory. Motion from B to A; from A′ to B′; from B′ to A′ and from A to B are S.H.M. guided by equation (3.4.1) and the motion from A to A′ and from A′ to A are uniform.

If t_2 be the time from A to O (or from A′ to O) then $t_2 = \frac{a}{v_2} = \frac{a}{b-a}\sqrt{\frac{am}{\lambda}}$.

Time from B to O is then $t_1 + t_2$.

Therefore the time period of oscillation is

T = $4(t_1 + t_2)$ (from symmetry)

$$= 4\left[\frac{\pi}{2}\sqrt{\frac{am}{\lambda}} + \frac{a}{b-a}\sqrt{\frac{am}{\lambda}}\right] = 2\sqrt{\frac{am}{\lambda}}\left[\pi + \frac{2a}{b-a}\right] \qquad \text{... (3.4.6).}$$

(ii) Oscillation in vertical position of a string.

Let OA = a be the natural length of the string which is fixed at the end O. Let a mass m be tied at the end A and it hangs in equilibrium at B. If T_0 be the tension of the string, the weight of the mas m, mg balances T_0. Thus

$$mg = T_0 = \lambda.\frac{AB}{a} \Rightarrow AB = \frac{mga}{\lambda}.$$

If now the mass m be pulled vertically downwards upto the position C such that OC = b and is then let go, the mass m will move upward from the position of instanteneous rest at C under the forces (i) tension T upward and (ii) $wt.\ mg$ downward (See fig. 3.4).

Let at time t, P be the position of the mass m where OP = x; the extension of the string is $x - a$ and tension T at P is $T = \lambda\frac{x-a}{a}$.

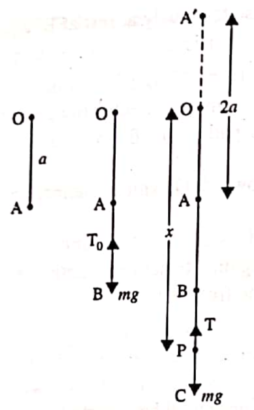

Fig. 3.4

Now equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - T \qquad \left(\frac{d^2x}{dt^2} > 0 \text{ in the direction } \overrightarrow{OC}\right)$$

$$\text{or, } \frac{d^2x}{dt^2} = g - \frac{\lambda}{am}(x-a) \text{ or, } \frac{d^2x}{dt^2} = -\frac{\lambda}{am}\left(x - a - \frac{mga}{\lambda}\right)$$

$$\text{or, } \frac{d^2y}{dt^2} = -\frac{\lambda}{am}y \qquad \text{... (3.4.7)}$$

where $y = x - a - \frac{mga}{\lambda}$. Equation (3.4.7) clearly shows that the mass m executes a simple harmonic oscillation with centre at $y = 0$ i.e. at $x = a + \frac{mga}{\lambda}$ = OB. i.e. at the point B, the equilibrium position of the mass.

From (3.4.7), multiplying both sides of it by $2\frac{dy}{dt}$ and integrating, we have

$$\left(\frac{dy}{dt}\right)^2 = c_1 - \frac{\lambda}{am}y^2 \text{ or, } \left(\frac{dx}{dt}\right)^2 = c_1 - \frac{\lambda}{am}\left(x - a - \frac{mga}{\lambda}\right)^2 \qquad \text{... (3.4.8)}$$

Initially, at $t = 0$, $x = b$, $\frac{dx}{dt} = 0$. Therefore,

$$c_1 = \frac{\lambda}{am}\left(b - a - \frac{mga}{\lambda}\right)^2$$

Hence, from(3.4.8), we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}\left[\left(b - a - \frac{mga}{\lambda}\right)^2 - \left(x - a - \frac{mga}{\lambda}\right)^2\right]$$

$$= \frac{\lambda}{am}(b - x)\left(x + b - 2\left(a + \frac{mga}{\lambda}\right)\right) \text{... (3.4.9)}$$

Case 1. Let $b < a + \frac{2mga}{\lambda}$. Then $\frac{dx}{dt} = 0$ again

When $x = 2\left(a + \frac{mga}{\lambda}\right) - b > a$ i.e. before reaching the point A the velocity of the mass will vanish and hence the mass will oscillate between $x = b$ and $x = 2\left(a + \frac{mga}{\lambda}\right) - b$ i.e. the mass will oscillate with centre at B up and down.

Case 2. Let $b > a + \frac{2mga}{\lambda}$. Then velocity of the mass on reaching the point A i.e. $x = a$, is given by $\left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}(b-a)\left(b - a - \frac{2mga}{\lambda}\right) > 0$

i.e., the mass m in its upward motion at A has non-zero velocity, but tension ceases to act as the string becomes unstretched. So the mass m moves vertically upward like a free body under the downward acceleration due to gravity.

The velocity of the mass m at A in its upward motion is

$$v_1 = -\sqrt{\frac{\lambda}{am}(b-a)\left(b - a - \frac{2mga}{\lambda}\right)}$$

(negative sign, since the mass m is moving in the direction of x decreasing)

As $\frac{2|v_1|}{g}$ is the maximum height the mass m can rise above A, so if $\frac{2|v_1|}{g} \le 2a$ (= AA'), A' is a point above O such that OA' = OA = a, then the mass after reaching its highest point (where the string still remains unstretched) will come down under gravity and shall reach the point A with velocity

$$+\sqrt{\frac{\lambda}{am}(b-a)\left(b - a - \frac{2mga}{\lambda}\right)} \text{ downwards.}$$

After that the string will be again extended and equation (3.4.7) holds again for the downward motion.

It can be seen that $\frac{dx}{dt}$ will vanish again at $x = b$ i.e. at C, with tension now acting upward at C. So the motion is then repeated. In this case also the motion is oscillatory.

If $b = a + \frac{2mga}{\lambda}$, the particle will oscillate between A and C, since from (3.4.9),

$$\left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}(b-x)(x-a) \Rightarrow \frac{dx}{dt} = 0 \text{ when } x = a \text{ and } x = b.$$

3.5. Composition of two simple harmonic motions along the same straight line.

(a) Time periods of two simple harmonic motions are same

Let a particle move along a straight line under two simple harmonic motions simultaneously. This situation can be visualised by considering a particle oscillating along a groove on a block of wood and the wood being made to oscillate along the same line simultaneously.

Let $\frac{2\pi}{n}$ be the equal time period of the two simple harmonic motions along the same straight line. With a common origin (the centre of mean position) on the line, let the displacements due to the two motions be

$x_1 = a_1 \cos(nt + \varepsilon_1)$ and $x_2 = a_2 \cos(nt + \varepsilon_2)$

where a_1, a_2 are amplitudes and $\varepsilon_1, \varepsilon_2$ are epochs of the two motions.

The resultant displacement due to two simultaneous motions is given by

$$x = x_1 + x_2 = a_1 \cos(nt + \varepsilon_1) + a_2 \cos(nt + \varepsilon_2)$$

$$= (a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2) \cos nt - (a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2) \sin nt \quad \ldots (3.5.1)$$

$$\left.\begin{aligned} \text{Let } & a_1 \cos\varepsilon_1 + a_2 \cos \varepsilon_2 = A \cos\varepsilon \\ \text{and } & a_2 \sin \varepsilon_1 + a_2 \sin\varepsilon_2 = A \sin\varepsilon \end{aligned}\right\} \quad \ldots (3.5.2)$$

where A, ε are constants.

Then using (3.5.2), we have from (3.5.1)

$$x = A \cos(nt + \varepsilon) \quad \ldots (3.5.3)$$

From (3.5.2), we have on squaring and adding,

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\varepsilon_1 - \varepsilon_2)$$

$$\text{or, } A = \sqrt{[a_1^2 + a_2^2 + 2a_1a_2 \cos(\varepsilon_1 - \varepsilon_2)]} \quad \ldots (3.5.4)$$

and on division,

$$\tan\varepsilon = \frac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}$$

$$\text{or, } \varepsilon = \tan^{-1}\left[\frac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}\right] \quad \ldots (3.5.5)$$

The resultant motion given by (3.5.3) is a S.H.M. of same time period as that of the two simultaneous motions along the same line, whose amplitude and epoch are respectively given by (3.5.4) and (3.5.5).

(b) Time periods of two simple harmonic motions are nearly equal

Let $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$ be the two nearly equal time periods of two simple harmonic motions along same straight line.

Let $n_2 - n_1 = \lambda$, which is very small.

With a common origin, let its displacements due to the two motions be given by

$$x_1 = a_1 \cos(n_1t + \varepsilon_1) \quad \text{and} \quad x_2 = a_2 \cos(n_2t + \varepsilon_2)$$

i.e. $x_2 = a_2 \cos(n_1t + \varepsilon')$ where $\varepsilon' = \lambda t + \varepsilon_2$.

Then the displacement due to the resultant motion is given by

$$x = x_1 + x_2 = a_1 \cos(n_1t + \varepsilon_1) + a_2 \cos(n_1t + \varepsilon')$$

$$= (a_1 \cos \varepsilon_1 + a_2 \cos\varepsilon') \cos n_1t - (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon') \sin n_1t$$

$$\text{i.e. } x = A \cos(n_1t + \varepsilon) \quad \ldots (3.5.6)$$

$$\left.\begin{aligned} \text{where } & A \cos\varepsilon = a_1\cos \varepsilon_1 + a_2 \cos \varepsilon' \\ & A \sin \varepsilon = a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon' \end{aligned}\right\} \quad \ldots (3.5.7)$$

Squaring and adding, we have from (3.5.7)

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\varepsilon' - \varepsilon_1)$$

$$\text{i.e., } A = \sqrt{\left[a_1^2 + a_2^2 + 2a_1a_2 \cos(\lambda t + \varepsilon_2 - \varepsilon_1)\right]} \quad \ldots (3.5.8)$$

$$\tan\varepsilon = \frac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon'}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon'}$$

$$\text{or, } \varepsilon = \tan^{-1} \frac{a_1 \sin\varepsilon_1 + a_2 \sin(\varepsilon_2 + \lambda t)}{a_1 \cos\varepsilon_1 + a_2 \cos(\varepsilon_2 + \lambda t)} \quad \ldots (3.5.9)$$

With change in time, λt changes and hence A and ε are not constants, but vary with time very slowly, since $\lambda = n_2 - n_1$ is very small. From (3.5.8), we see that A is maximum when $\lambda t + \varepsilon_2 - \varepsilon_1$ is an even multiple of π and maximum value of A is $a_1 + a_2$ and A is minimum when $\lambda t + \varepsilon_2 - \varepsilon_1$ is an odd multiple of π and minimum value of A is $a_1 \sim a_2$.

Thus resultant of two simple harmonic motions with nearly equal time period along the same straight line is simple harmonic at any instant of time, with time-period approximately equal to either of the two component motions, the amplitude and epoch of the resultant S.H.M slowly vary with time; amplitude A varies from a definite minimum value $a_1 \sim a_2$ to a definite maximum value $a_1 + a_2$, the periodic time of this change is $\frac{2\pi}{\lambda}$ i.e. $\frac{2\pi}{n_2 - n_1}$.

Remark **:** To cite an example of two S.H.M. with nearly equal time periods along the same straight line, we see that the composition of two vibrations with nearly equal periods gives rise to the phenomenon of Beats which is actually a sound with a variable amplitude varying from a low value to a high value periodically.

Note : Composition of two S.H.M. of two unequal time-periods can not be made possible.

3.6. Damped Harmonic Oscillations

We now consider the motion of a particle in a straight line under a controlling force which is always directed to a fixed point on the line and is proportional to its distance from the fixed point, in a medium which offers a small resistance to its motion. Let us investigate the effect of force of resistance on the motion.

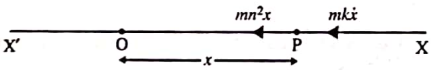

Fig. 3.5 (direction of motion along $\overrightarrow{OX}$)

Let at any time t, P be the position of a particle of mass m at a distance x (= OP) from the fixed point O. The particle is moving in the direction $\overrightarrow{OX}$, so that velocity $\dot{x}\left(=\frac{dx}{dt}\right)$ of the particle at P is positive in the direction $\overrightarrow{OX}$. Then the controlling force on the particle at P is mn^2x directed towards O and the resistance of the medium to the motion is $mk\dot{x}$ along $\overrightarrow{XO}$, where k is a small positive number.

Thus the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -mn^2x - mk\frac{dx}{dt} \qquad \text{... (3.6.1)}$$

acceleration $\frac{d^2x}{dt^2}$ is positive along $\overrightarrow{OX}$ (See fig. 3.5).

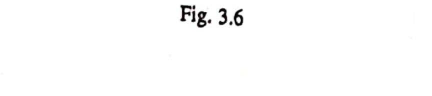

Fig. 3.6

If at time t, the position of the particle is at P where OP = x and the particle is moving in the direction $\overrightarrow{XO}$ i.e. in the direction of x decreasing then $\dot{x} < 0$, and resistance of the medium to the motion is in the direction $\overrightarrow{OX}$ i.e. resistance acts in the direction of x increasing. Then the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -mn^2x - mk\frac{dx}{dt}, \text{ (resistance } - mk\frac{dx}{dt} > 0 \text{ in the direction } \overrightarrow{OX})$$

which is same as the equation (3.6.1) (See fig. 3.6).

Thus the equation (3.6.1) is the equation of motion for all positions P on the right of O, irrespective of the direction in which the particle is moving.

Similarly, it can be seen that the equation (3.6.1) is the equation of motion for all positions P on the left of O, which is indiependent of the direction in which the particle is moving.

Now the equation (3.6.1) can be written in the form

$$(D^2 + kD + n^2)x = 0 \qquad \text{... (3.6.2)}$$

where $D \equiv \frac{d}{dt}$, which is a second order linear homogeneous differential equation with constant coefficients.

The auxiliary equation is $\alpha^2 + k\alpha + n^2 = 0$. Let its roots be α_1, α_2.

$$\text{Then } \alpha_1 = \frac{-k - \sqrt{k^2 - 4n^2}}{2}, \ \alpha_2 = \frac{-k + \sqrt{k^2 - 4n^2}}{2} \qquad \text{... (3.6.3)}$$

Nature of the roots α_1, α_2 depend on the value of $k^2 - 4n^2$. We consider three different cases.

Case 1. $k^2 - 4n^2 < 0$. Both α_1, α_2 are then complex numbers.

Let $k^2 - 4n^2 = -4\lambda^2$ ($\lambda \neq 0$).

Then from (3.6.3), we have $\alpha_1 = -\frac{k}{2} + i\lambda$, $\alpha_2 = -\frac{k}{2} - i\lambda$ (where $i^2 = -1$). The general solution of equation (3.6.2) is

$$x = e^{-\frac{1}{2}kt}\,(A\cos\lambda t + B\sin\lambda t) \qquad \text{... (3.6.4)}$$

The constants A, B are to be determined from initial conditions of motion.

The solution (3.6.4) can also be put in the form

$$x = c\,e^{-\frac{1}{2}kt}\cos(\lambda t + \varepsilon) \qquad \text{... (3.6.5),}$$

by taking $A = c\cos\varepsilon$ and $B = -c\sin\varepsilon$, where c, ε are arbitrary constants to be determined by using initial conditions of motion.

From (3.6.5) we observe that the resulting motion is not exactly a S.H.M. but is almost a simple harmonic motion of time-period $\left(= \frac{2\pi}{\sqrt{n^2 - \frac{1}{4}k^2}}\right) \simeq \frac{2\pi}{n}$, (since

k is very small), so that time period of the resulting motion is nearly equal to the time-period of natural S.H.M. (when there is no resistance). The amplitude of the motion is $ce^{-\frac{1}{2}kt}$, which steadily decays with time and when t is very large, amplitude tends to zero. Such a motion of diminishing amplitude is called damped harmonic oscillation, the quantity k measures its damping and the quantity $e^{-\frac{1}{2}kt}$ is called damping coefficient.

The effect of a small resistance of the medium on the particle executing a natural S.H.M. of time-period $\frac{2\pi}{n}$ is that it increases the time period slightly as $\lambda < n$, and it gradually decreases the amplitude and after a long time the motion ultimately dies out. The motion is said to be **underdamped**.

Case 2. $k^2 - 4n^2 > 0$. Then the roots α_1, α_2 of the auxiliary equation are real and distinct.

Taking $k^2 - 4n^2 = 4\lambda^2$ $(\lambda \neq 0)$, we have $\alpha_1 = -\frac{1}{2}k + \lambda$, $\alpha_2 = -\frac{1}{2}k - \lambda$.

The general solution of equation (3.6.2) is

$$x = e^{-\frac{1}{2}kt}\,[A' e^{\lambda t} + B' e^{-\lambda t}] \qquad \text{... (3.6.6)}$$

A′, B′ are arbitrary constants to be determined from initial conditions.

From (3.6.6), it is obvious that the motion is non-oscillatory and it ultimately dies out, since $x \to 0$ as $t \to \infty$ (as $k^2 - 4\lambda^2 > 0$).

In this case the motion is said to be **overdamped**.

Case 3. $k^2 - 4n^2 = 0$. Then α_1, α_2 are two real equal roots of the auxiliary equation and $\alpha_1 = \alpha_2 = -\frac{1}{2}k$.

The general solution of the equation (3.6.2) is

$$x = (c_1 + c_2 t)\,e^{-\frac{1}{2}kt} \qquad \text{... (3.6.7)}$$

c_1, c_2 are arbitrary constants. Here also the motion is non-oscillatory. As $x \to 0$ when $t \to \infty$, the motion ultimately dies out. The motion in this case is said to be **critically damped**.

3.7. Forced Oscillation

A particle moves in a straight line under the action of a controlling force which is always directed to a fixed point on the line and is proportional to its distance from the fixed point and is simultaneously acted on by a periodic disturbing force of magnitude F cos pt per unit mass of the particle. Let us investigate the effect of disturbing force on the natural S.H.M.

Let at time t, P be the position of a particle of mass m moving on a straight line X′OX under a controlling force mn^2x directed towards O, a fixed point on the line, where OP $= x$.

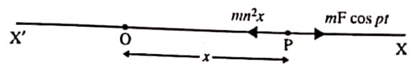

Fig. 3.7

At the same instant t, the particle is also acted upon by a periodic force $mF\cos pt$ in the direction of motion i.e. along $\overrightarrow{OX}$ (see fig. 3.7).

The equation of motion is $m\dfrac{d^2x}{dt^2} = -mn^2x + mF\cos pt$

or, $$(D^2 + n^2)x = F\cos pt \qquad \text{... (3.7.1)}$$

where $D \equiv \dfrac{d}{dt}$.

To solve the equation (3.7.1), let $x = e^{\alpha t}(\neq 0)$ be a solution of $(D^2 + n^2)x = 0$. Then the auxiliary equation is $\alpha^2 + n^2 = 0$, whose roots are $\alpha = \pm\, i\,n (i^2 = -1)$. So the complementary function is $x_1 = A\cos(nt + \varepsilon)$ where A, ε are arbitrary constants. The particular integral is $x_2 = \dfrac{1}{D^2 + n^2}F\cos pt = \dfrac{F\cos pt}{n^2 - p^2}$, where we assume that $p \neq n$.

Then the complete primitive of the equation (3.7.1) is

$$x = x_1 + x_2 = A\cos(nt + \varepsilon) + \frac{F\cos pt}{n^2 - p^2} \quad (p \neq n) \qquad \text{... (3.7.2),}$$

we see that the motion of the particle is a composition of (1) an undisturbed (or natural) S.H.M. of period $\frac{2\pi}{n}$, whose amplitude A depends on the initial conditions of motion and (2) an oscillatory motion of period $\frac{2\pi}{p}$ and amplitude $\dfrac{F}{n^2 - p^2}$ which is independent of initial conditions of motion.

The former motion is called free oscillation and the latter motion is a superimposed S.H.M and is called forced oscillation, whose time-period is same as the period of disturbing force. The amplitude of the foced oscillation depends on the period of forced oscillation.

When $p \simeq n$, the forcd oscillation represented by $\dfrac{F}{n^2 - p^2}\cos pt$, with a very large amplitude, will be dominating over the free oscillation represented by $A\cos(nt + \varepsilon)$ and the particle will oscillate with a period which is practically same as that of the forced oscillation.

If $p = n$, then the particular integral of the equation (3.7.1) is

$$\frac{1}{D^2 + n^2}\,F\cos nt = \text{Real part of F.}\ \frac{1}{D^2 + n^2}\,e^{int}$$

Now $\frac{1}{D^2+n^2}e^{int} = e^{int}\frac{1}{(D+in)^2+n^2}(1)$

$= e^{int}\frac{1}{D^2+2inD}(1) = e^{int}\frac{1}{2inD}\left(1+\frac{D}{2in}\right)^{-1}(1)$

$= e^{int}\frac{1}{2inD}(1) = \frac{-ie^{int}}{2n}t.$

Hence the particular integral is $\frac{F}{2n}t\sin nt$.

Thus the general solution of (3.7.1) is $x = A\cos(nt+\varepsilon) + \frac{F}{2n}t\sin nt$ ... (3.7.3)

From (3.7.3), we observe that amplitude of the forced oscillation continuously increases with time. This resulting motion is called resonance and this happens when natural frequency is equal to the frequency of the disturbing periodic force.

3.8. Damped Forced Oscillation

We now consider the motion of a particle in a straight line under the action of a controlling force proportional to its distance from a fixed point on the line and directed towards the fixed point, in a medium which offers resistance proportional to its velocity and is simultaneously acted on by a periodic disturbing force of magnitude F cos pt per unit mass of the particle. We shall investigate the effect of force of resistance and periodic force on the free oscillation.

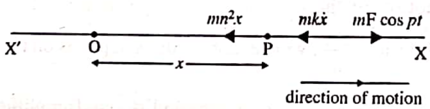

Fig. 3.8

Let at time t, P be the position of a particle of mass m moving on a straight line X'OX under the action of

(i) a controlling force mn^2x directed towards O, a fixed point on the line, where $x =$ OP,

(ii) a force of resistance $mk\dot{x}$ ($\dot{x}$ is the velocity at P) directed towards O, assuming that the particle is moving along $\overrightarrow{OX}$ (See fig. 3.8), and (ii) a disturbing periodic force of magnitude in $F\cos pt$, along $\overrightarrow{OX}$ (See fig. 3.8)

Then the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -mn^2x - mk\frac{dx}{dt} + mF\cos pt \text{ or, } (D^2+kD+n^2)x = F\cos pt \quad ... (3.8.1)$$

where $D \equiv \frac{d}{dt}$.

Let $x = e^{\alpha t}$ ($\neq 0$) be a solution of $(D^2 + kD + n^2)x = 0$, then the auxiliary equation is

$$\alpha^2 + k\alpha + n^2 = 0 \Rightarrow \alpha = \frac{1}{2}\left[-k \pm \sqrt{k^2-4n^2}\right].$$

Let $k < 2n$. Then $\alpha = \frac{1}{2}\left[-k \pm 2i\sqrt{n^2-\frac{k^2}{4}}\right]$ ($i^2 = -1$) and the complementary function is then $x_1 = Ae^{-\frac{1}{2}kt}\cos(\lambda t+\varepsilon)$, where $\lambda = \sqrt{n^2-\frac{k^2}{4}}$ and A, ε are arbitrary constants to be determined from initial conditions of motion.

The particular integral is then

$$x_2 = \frac{1}{D^2+kD+n^2}F\cos pt = F\frac{D^2+n^2-kD}{(D^2+n^2)^2-k^2D^2}\cos pt$$

$$= F\frac{(n^2-p^2)\cos pt + pk\sin pt}{(n^2-p^2)^2+p^2k^2} = \frac{F}{\sqrt{(n^2-p^2)^2+p^2k^2}}\cos(pt-\varepsilon')$$

where we assume that $\frac{n^2-p^2}{\sqrt{(n^2-p^2)^2+p^2k^2}} = \cos\varepsilon'$

and $\frac{pk}{\sqrt{(n^2-p^2)^2+p^2k^2}} = \sin\varepsilon'$ i.e. $\varepsilon' = \tan^{-1}\left(\frac{pk}{n^2-p^2}\right)$.

The general solution of the equation (3.8.1) is then

$$x = x_1 + x_2 = Ae^{-\frac{1}{2}kt}\cos(\lambda t+\varepsilon) + \frac{F}{\sqrt{(n^2-p^2)^2+p^2k^2}}\cos(pt-\varepsilon') \quad ... (3.8.2)$$

The resultant motion is then composition of two oscillations. The first part $Ae^{-\frac{1}{2}kt}\cos(\lambda t+\varepsilon)$ corresponds to damped harmonic oscillation (without the presence of disturbing force) with light damping, since $k < 2n$ having period $\frac{2\pi}{\lambda}$ and the second part $\frac{F}{\sqrt{(n^2-p^2)^2+p^2k^2}}\cos(pt-\varepsilon')$ is a super imposed S.H.M, called forced oscillation.

As t increases, the amplitude of the damped harmonic oscillation continually diminishes and ultimately dies out when $t \to \infty$. Thus after a long time the resulting motion tends to a forced oscillation of amplitude $\dfrac{F}{\sqrt{(n^2-p^2)^2+p^2k^2}}$ and of period equal to the period of the disturbing force.

Remark : If $p = n$ i.e., the period of the disturbing force is equal to the period of forced oscillation then the particular integral part of the general solution of (3.8.1) is $x_2 = \dfrac{F}{kn}\sin nt$, which corresponds to a forced oscillation having maximum amplitude $\dfrac{F}{kn}$. For small k, the forced oscillation has a very large amplitude however small F may be. Hence a small periodic force may produce an oscillation of very large amplitude, if its period is nearly equal to the period of free oscillation.

3.9. Illustrative Solved Examples

***Example* 3.9.1.** A particle performing a S.H.M. of period T about a centre O passes through a point P with a velocity v in the direction $\overrightarrow{OP}$. If OP be equal to x and the particle returns to P in time t then show that $t = \dfrac{T}{\pi}\tan^{-1}\dfrac{vT}{2\pi x}$.

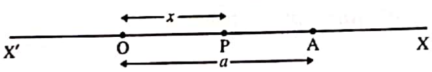

Fig. 3.9

Solution : Let a = OA be the amplitude of the S.H.M having time-period $T = \dfrac{2\pi}{\sqrt{\mu}}$. Then velocity $\dot{x}\,(= v)$ and distance x of the particle executing S.H.M. are related by

$$v^2 = \mu(a^2 - x^2) \qquad \text{... (1)}$$

Since the particle is at P in its motion along $\overrightarrow{OP}$, so $v > 0$, hence

$$v = \frac{dx}{dt} = +\sqrt{\mu}\sqrt{a^2-x^2} \qquad \text{... (2)}$$

$$\Rightarrow \int_{x=x}^{a}\frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\mu}\int_{t=0}^{t_1}dt$$

(we are measuring time from the instant the particle is at P in its motion along $\overrightarrow{OA}$ and t_1 is the time from P to A (see fig. 3.9))

$$\Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\sin^{-1}\frac{x}{a}\right]_x^a = \frac{1}{\sqrt{\mu}}\left[\frac{\pi}{2}-\sin^{-1}\frac{x}{a}\right] = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}.$$

From symmetry of oscillatory motion, the time from A to P in its motion along $\overrightarrow{AO}$ is equal to the time from P to A.

Since t is the time taken by the particle to return to P, so $t = 2t_1 = \dfrac{2}{\sqrt{\mu}}\cos^{-1}\dfrac{x}{a}$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\quad\left(\because \cos^{-1}y = \tan^{-1}\frac{\sqrt{1-y^2}}{y}\right)$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi x}\right)\ \text{(using (2))}.$$

***Example* 3.9.2.** A particle rests in equilibrium under the attraction of the centres of force which attract directly as the distance, their attractions per unit mass at unit distance being μ and μ'. The particle is displaced towards one of them. Show that its motion is oscillatory of period $\dfrac{2\pi}{\sqrt{\mu+\mu'}}$.

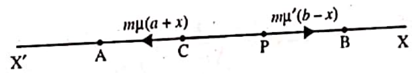

Fig. 3.10

Solution : Let a particle of mass m rest in equilibrium at C under the attraction of two centres of force A and B where AC = a, CB = b. If $\mu\,(> 0)$ and $\mu'\,(> 0)$ be the attractions per unit mass at unit distance then for equilibrium position at C, $\mu a = \mu' b$ ... (1) (See fig. 3.10)

Let at time t, P be the position of the particle where, the particle is displaced at a distance $x(=$ CP) towards B. Then the forces acting on the particle at P are $m\mu(a + \text{CP}) = m\mu(a + x)$ acting towards A and $m\mu'(b - \text{CP}) = m\mu(b - x)$ acting towards B (see fig. 3.10). Since acceleration is positive in the direction of x increasing, the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -m\mu(a+x) + m\mu'(b-x) \text{ or, } \frac{d^2x}{dt^2} = -(\mu+\mu')x \ \text{(by (1))}.$$

which shows that the motion of the particle is a S.H.M. having time period $\dfrac{2\pi}{\sqrt{\mu+\mu'}}$.

***Example* 3.9.3.** A particle of mass m rests on a smooth horizontal plane and is attached to one end of a light elastic string, the other end of which is fastened to a fixed point on the plane. The unstretched length of the string being l, show that, if the particle be moved along the plane until its distance from the fixed point is $l'\,(> l)$, and is then let go, it will pass the fixed point after a time given by $t = \sqrt{\dfrac{ml}{\lambda}}\left(\dfrac{\pi}{2}+\dfrac{l}{l'-l}\right)$.

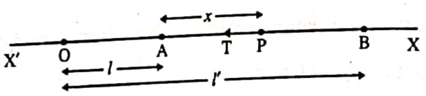

Fig. 3.11

Solution : Let OA = l be natural length of a light elastic string, which is attached to a fixed point O on the plane. The free end A is attached with a particle of mass m. The particle is pulled along the plane to point B such that OB = $l'(> l)$ and is then let go. Then the particle will move along $\overrightarrow{BO}$ under the action of tension of the string directed towards O (See fig. 3.11).

Let at time t, P be the position of the particle such that AP = x. There x is the extension of the string and tension at P is $T = \lambda \frac{x}{l}$.

Then equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -T \text{ or, } \frac{d^2x}{dt^2} = -\frac{\lambda}{ml}x \qquad \ldots (1)$$

The general solution of equation (1) is

$$x = c\cos\left(\sqrt{\frac{\lambda}{ml}}t + \varepsilon\right) \qquad \ldots (2)$$

where c, ε are constants to be determined from initial conditions of motion.

From (2), $$\frac{dx}{dt} = -c\sqrt{\frac{\lambda}{ml}}\sin\left(\sqrt{\frac{\lambda}{ml}}t + \varepsilon\right) \qquad \ldots (3)$$

Initially at $t = 0$, $x = l' - l$, $\frac{dx}{dt} = 0$.

Then from (2) and (3), we have

$l' - l = c\cos\varepsilon$ and $0 = \sin\varepsilon \Rightarrow \varepsilon = 0$ and $c = l' - l$.

Hence from (2), the distance of the particle from A at time t is given by

$$x = (l' - l)\cos\left(\sqrt{\frac{\lambda}{ml}}\cdot t\right) \qquad \ldots (4)$$

and from (3), $$\frac{dx}{dt} = -(l' - l)\sqrt{\frac{\lambda}{ml}}\sin\left(\sqrt{\frac{\lambda}{ml}}t\right) \qquad \ldots (4)$$

If t_1 be the time from B to A then at $t = t_1$, $x = 0$. So, from (4), we have

$$\cos\left(\sqrt{\frac{\lambda}{ml}}\cdot t_1\right) = 0 \Rightarrow t_1 = \frac{\pi}{2}\sqrt{\frac{ml}{\lambda}}.$$

At $t = t_1$, the particle reaches the point A with velocity $v = (l' - l)\sqrt{\frac{\lambda}{ml}}$ directed along $\overrightarrow{AO}$, which is obtained from (4) for $t = t_1$.

At A, the string is unstretched and hence tension ceases to act. Thus the particle will move with uniform velocity v and it will reach the fixed point O in time t_2 where

$$vt_2 = OA = l. \Rightarrow t_2 = \frac{l}{l' - l}\sqrt{\frac{ml}{\lambda}}.$$

Thus the particle will reach the point O starting from B in time t where

$$t = t_1 + t_2 = \sqrt{\frac{ml}{\lambda}}\left[\frac{\pi}{2} + \frac{l}{l' - l}\right].$$

***Example* 3.9.4.** A particle of unit mass is tied by four equal elastic strings of natural length l and modulus of elasticity λ to the corners of a square. If the particle is displaced a small distance towards one of the corners and then set free, prove that the time of a small oscillation is $\pi\sqrt{\frac{al}{\lambda(a-l)}}$, where a is the length of the diagonal of the square and a is so much greater than l that the strings remain unstretched.

Solution : Let ABCD be a square. AC, BD are its two diagonals of length a each.

A particle of unit mass is tied by four equal elastic strings of natural length l and modulus of elasticity λ to the corners A, B, C, D of the square.

Then O, the centre of the square is the position of equilibrium of the particle.

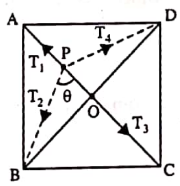

Fig. 3.12

Suppose the particle at O be displaced through a small distance x towards the corner A and now P be its position so that OP = x.

Then the forces acting at P are tension T_1 along $\overrightarrow{PA}$, tension T_2 along $\overrightarrow{PB}$, tension T_3 along $\overrightarrow{PC}$ and tension T_4 along $\overrightarrow{PD}$.

From fig. 3.12, PA = OA – OP = $\frac{a}{2} - x$

$$PB = \sqrt{OP^2 + OB^2} = \sqrt{x^2 + \frac{a^2}{4}} = PD \text{ and } PC = OP + OC = x + \frac{a}{2}.$$

By Hooke's law,

$$T_1 = \lambda\frac{\frac{a}{2} - x - l}{l}, \quad T_2 = \lambda\frac{\sqrt{x^2 + \frac{a^2}{4}} - l}{l} = T_4 \text{ and } T_3 = \lambda\frac{\frac{a}{2} + x - l}{l}.$$

The equation of motion of the particle is then

$$\frac{d^2x}{dt^2} = T_1 - T_3 - T_2\cos\theta - T_4\cos\theta \quad \dots(1)$$

where $\theta = \angle OPB$ and $\cos\theta = \dfrac{OP}{PB} = \dfrac{x}{\sqrt{x^2 + \frac{a^2}{4}}}$.

From (1), we have

$$\frac{d^2x}{dt^2} = \lambda\frac{\frac{a}{2} - x - l}{l} - \lambda\frac{\frac{a}{2} + x - l}{l} - 2\lambda\frac{\sqrt{x^2 + \frac{a^2}{4}} - l}{l} \cdot \frac{x}{\sqrt{x^2 + \frac{a^2}{4}}}$$

$$\text{or, } \frac{d^2x}{dt^2} = -\frac{2\lambda}{l}x - \frac{2\lambda}{l}x\left(1 - \frac{l}{\sqrt{x^2 + \frac{a^2}{4}}}\right)$$

$$\text{or, } \frac{d^2x}{dt^2} = -\frac{4\lambda}{l}x + \frac{2\lambda x}{\frac{a}{2}}\left(1 + \frac{4x^2}{a^2}\right)^{-\frac{1}{2}}.$$

Since x is very small, we neglect square and higher powers of x. Thus we have

$$\frac{d^2x}{dt^2} = -\frac{4\lambda}{al}(a - l)x \quad \dots(2)$$

From (2), we see that the motion of the particle is a S.H.M. having time-period

$$T = \frac{2\pi}{\sqrt{\frac{4\lambda}{al}(a-l)}} = \pi\sqrt{\frac{al}{\lambda(a-l)}}.$$

Example 3.9.5. A particle is moving in a S.H.M. of amplitude a and period T and when in a position of instantaneous rest is given a blow which imparts a velocity u towards the mean position. Show that it will avrrive at its next position of instantaneous rest at a time less by $\frac{T}{\pi}\tan^{-1}\left(\frac{uT}{2\pi a}\right)$ than if it had not received the impulse. Show that it will continue in S.H.M of the same period but of amplitude $\left(a^2 + \frac{u^2T^2}{4\pi^2}\right)^{\frac{1}{2}}$.

Solution : Let at time t, P be the position of a particle of mass m moving along the line X'OX in a simple harmonic motion about the centre O with time period $T = \frac{2\pi}{\sqrt{\mu}}$. Let a be the amplitude of the S.H.M. and A, A′ be the two positions of

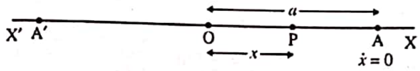

Fig. 3.13

instantaneous rest, where OA = a (see fig. 3.13). If OP = x, then the equation of the motion is

$$m\frac{d^2x}{dt^2} = -m\mu x \quad \dots(1)$$

Time from A to A′ is $\frac{1}{2}T = \frac{\pi}{\sqrt{\mu}}$.

When the particle receives a blow at A, its equation of motion is

$$mv\frac{dv}{dx} = -m\mu x \quad \dots(2)$$

where v is velocity at P.

From (2) $v\,dv = -\mu x\,dx$. On integration we have $v^2 = c_1 - \mu x^2$. When $x = a$, i.e. at A, $v = u$(given). So $c_1 = u^2 + \mu a^2$. Thus we have

$$v^2 = u^2 + \mu a^2 - \mu x^2 \quad \dots(3)$$

The velocity will be zero again when

$$\mu x^2 = u^2 + \mu a^2 \text{ or, } x = \sqrt{a^2 + \frac{u^2}{\mu}} = \sqrt{a^2 + \frac{u^2T^2}{4\pi^2}} \quad \left(\because T = \frac{2\pi}{\sqrt{\mu}}\right).$$

Thus in the second case, the amplitude of the S.H.M. will be $\left(a^2 + \frac{u^2T^2}{4\pi^2}\right)^{\frac{1}{2}}$.

Let t_1 be the time from $x = a$ to $x = -a$, i.e. from A to A′.

Then from (3), we have $\frac{dx}{dt} = -\sqrt{u^2 + \mu a^2 - \mu x^2}$

(negative sign, since the particle is moving in the direction of x decreasing)

$$\text{or, } \sqrt{\mu}\int_0^{t_1} dt = -\int_{x=a}^{-a}\frac{dx}{\sqrt{a^2 + \frac{u^2}{\mu} - x^2}}$$

$$\Rightarrow \sqrt{\mu}\,t_1 = 2\int_0^a \frac{dx}{\sqrt{a^2 + \frac{u^2}{\mu} - x^2}} = 2\left[\sin^{-1}\frac{x}{\sqrt{a^2 + \frac{u^2}{\mu}}}\right]_0^a$$

$$= 2\sin^{-1}\frac{a}{\sqrt{a^2 + \frac{u^2}{\mu}}} = 2\tan^{-1}\frac{a}{\frac{u}{\sqrt{\mu}}} \quad \left(\because \sin^{-1}z = \tan^{-1}\frac{z}{\sqrt{1-z^2}}\right)$$

$$\therefore \frac{2\pi}{T} t_1 = 2\tan^{-1}\frac{a\sqrt{\mu}}{u} \quad \text{or, } t_1 = \frac{T}{\pi}\tan^{-1}\frac{2a\pi}{uT}.$$

Hence the required diffrence in time is

$$\frac{T}{2} - t_1 = \frac{T}{2} - \frac{T}{\pi}\tan^{-1}\frac{2\pi a}{uT} = \frac{T}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{2\pi a}{uT}\right]$$

$$= \frac{T}{\pi}\cot^{-1}\frac{2\pi a}{uT} = \frac{T}{\pi}\tan^{-1}\left(\frac{uT}{2\pi a}\right).$$

***Example* 3.9.6.** A light elastic string AB of length a is fixed at A and is such that if a weight W be attached to B the string will be stretched to double its length. If a weight $\frac{1}{4}$W be fastened to B and let fall from A, prove that

(i) the subsequent motion is simple harmonic,

(ii) its amplitude is $\frac{3}{4}a$,

(iii) the distance through which it will fall is $2a$, and

(iv) the period of oscillation is $\frac{1}{2}\sqrt{\frac{l}{g}}(4\sqrt{2} + \pi + 2\sin^{-1}\frac{1}{3})$.

***Solution* :** If λ be the modulus of elasticity of a light elastic string of natural length AB $= a$, and a wt. W, if attached to the free and B of the string, stretches the string to double its length, AA′ (= 2AB) then tension at A′ will belance the *wt.* W for equilibrium, so that $\lambda\frac{2a-a}{a} = W \Rightarrow \lambda = W$.

If a weight $\frac{1}{4}$W is now attached to the end B and is let fall from the point A, then the *wt.* $\frac{1}{4}$W will reach the point B with a speed $u = \sqrt{2ga}$ and the string is then stretched. Suppose, at time t, P be the position of the *wt.* $\frac{1}{4}$W where BP $= x$. The forces acting at P are tension $T = \lambda\frac{x}{a} = \frac{W}{a}x$ in vertically upward direction and the *wt.* $\frac{1}{4}$W in vertically downward direction i.e. in the direction of motion (see fig. 3.14).

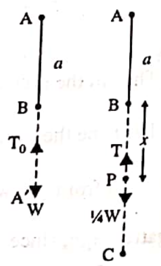

Fig. 3.14

Then equation of motion of the *wt.* $\frac{1}{4}$W is

$$\frac{1}{4}\frac{W}{g}\frac{d^2x}{dt^2} = \frac{1}{4}W - \frac{W}{a}x$$

$$\text{or, } \frac{d^2x}{dt^2} = -\frac{4g}{a}\left(x - \frac{a}{4}\right) \quad \text{or, } \frac{d^2y}{dt^2} = -\frac{4g}{a}.y \qquad \text{... (1)}$$

where $y = x - \frac{a}{4}$.

Equation (1) shows that the subsequent motion of the *wt.* $\frac{1}{4}$W is simple harmonic about the centre $y = 0$ i.e. $x = \frac{a}{4}$.

From (1), multiplying both sides by $2\frac{dy}{dt}$ and integrating, we have

$$\left(\frac{dy}{dt}\right)^2 = c_1 - \frac{4g}{a}y^2 \qquad \text{... (2)}$$

At B i.e., at $x = 0$, $y = -\frac{a}{4}, \frac{dy}{dt} = \sqrt{2ga}$.

Hence from (2), we have $c_1 = 2ga + \frac{ag}{4} = \frac{9ag}{4}$

Thus from (2), we have

$$\left(\frac{dy}{dt}\right)^2 = \frac{9ag}{4} - \frac{4g}{a}y^2 \qquad \text{... (3)}$$

The velocity of the particle will vanish

i.e. $\frac{dy}{dt}\left(= \frac{dx}{dt}\right) = 0$, where $\frac{9ag}{4} - \frac{4g}{a}y^2 = 0$ i.e. when $y = \pm\frac{3a}{4}$, i.e., when $x = a$ and $-\frac{a}{2}$.

Since $x = \frac{a}{4}$ is the centre of oscillation, hence the greatest displacement on either side of the centre of oscillation is $a - \frac{a}{4} = \frac{3a}{4}$, which is the amplitude of the S.H.M.

Since the velocity of the *wt.* $\frac{1}{4}$W in its downward motion vanishes at $x = a$, hence the distance through which it falls is AB$(= a) + a = 2a$.

From (3), we have

$$\frac{dy}{dt} = +2\sqrt{\frac{g}{a}}\sqrt{\left(\frac{3a}{4}\right)^2 - y^2} \qquad \text{... (4)}$$

(+ve sign is taken, since y increases in the direction of motion.

If t_1 be the time from A to B then $t_1 = \frac{u}{g} = \frac{\sqrt{2ag}}{g} = \sqrt{\frac{2a}{g}}$.

If t_2 be the time form B, where $y = -\frac{a}{4}$ to the point, say C, where $y = \frac{3a}{4}$ (i.e. $x = a$) then, form (4)

$$\int_0^{t_2} dt = \frac{1}{2}\sqrt{\frac{a}{g}} \int_{y=-\frac{a}{4}}^{\frac{3a}{4}} \frac{dy}{\sqrt{\left(\frac{3a}{4}\right)^2 - y^2}}$$

$$\Rightarrow t_2 = \frac{1}{2}\sqrt{\frac{a}{g}}\left[\sin^{-1}\frac{4y}{3a}\right]_{-\frac{a}{4}}^{\frac{3a}{4}} = \frac{1}{2}\sqrt{\frac{a}{g}}\left[\frac{\pi}{2} + \sin^{-1}\frac{1}{3}\right].$$

Thus time from A to C is $t_1 + t_2$. Hence total time of oscillations i.e. time from A to C and back to A is $2(t_1 + t_2)$ (from symmetry)

$$= 2\sqrt{\frac{2a}{g}} + \frac{1}{2}\sqrt{\frac{a}{g}}\left[\pi + 2\sin^{-1}\frac{1}{3}\right] = \frac{1}{2}\sqrt{\frac{a}{g}}\left[4\sqrt{2} + \pi + 2\sin^{-1}\frac{1}{3}\right].$$

***Example* 3.9.7.** A heavy particle is attached to the lower end of an elastic string, the upper end of which is fixed. The modulus of elasticity of the string is k times the weight of the particle. The mass is drawn vertically downwards until the length of the string becomes $(n + 1)$ times its original length and is then released. Show that the string will be just slack after a time $\sqrt{\frac{a}{kg}}\left[\pi - \cos^{-1}\frac{1}{nk-1}\right]$.

Solution : Let OA($= a$) be an untstretched length of a string, the upper end O of the string is fixed and a heavy particle of mass m is attached to the lower end A. The mass m is drawn vertically downwards upto the point B such that OB $= (n + 1)$ OA $= (n + 1)a$. Thus AB $= na$ and then the particle is released from B. Naturally, the particle will begin to move vertically upwards from rest at B under the action of tension in upward direction and weight of the porticle in downward direction (see fig. 3.15).

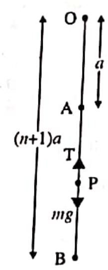

Fig. 3.15

Let at time t, P be the position of the particle, where AP $= x$. Then tension at P is $T = \lambda\frac{x}{a} = kmg.\frac{x}{a}$ ($\because \lambda = k\,mg$)

The equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - T \text{ or, } \frac{d^2x}{dt^2} = g - \frac{kg}{a}x \text{ or, } \frac{d^2y}{dt^2} = -\frac{kg}{a}y \qquad \text{... (1)}$$

where $y = x - \frac{a}{k}$.

Multiplying equation (1) $2\frac{dy}{dt}$ and integrating

we have $\left(\frac{dy}{dt}\right)^2 = c_1 - \frac{kg}{a}y^2$ ... (2)

At B, $x = na$, and hence $y = na - \frac{a}{k}$ and $\frac{dx}{dt} = \frac{dy}{dt} = 0$. Hence from (2), we obtain

$$c_1 = \frac{kg}{a}\left(na - \frac{a}{k}\right)^2 = \frac{ag}{k}(nk-1)^2.$$

Thus from (2), we have

$$\left(\frac{dy}{dt}\right)^2 = \frac{ag}{k}(nk-1)^2 - \frac{kg}{a}y^2 \text{ or, } \left(\frac{dy}{dt}\right)^2 = \frac{kg}{a}\left[\frac{a^2}{k^2}(nk-1)^2 - y^2\right]$$

$$\text{or, } \frac{dy}{dt} = -\sqrt{\frac{kg}{a}}.\sqrt{\frac{a^2}{k^2}(nk-1)^2 - y^2} \qquad \text{... (3)}$$

(negative sign is taken, since $\frac{dx}{dt}\left(=\frac{dy}{dt}\right)$ is + ve in the direction of x increasing, while in this case x is decreasing).

The string becomes just slack when the particle reaches the point A i.e., when $x = 0$. If t_1 be the required time from B to A i.e. from $x = na$ to $x = 0$, then, from (3), we have

$$\int_0^{t_1} dt = \sqrt{\frac{a}{kg}} \int_{y=na-\frac{a}{k}}^{0-\frac{a}{k}} \frac{-dy}{\sqrt{\frac{a^2}{k^2}(nk-1)^2 - y^2}} \left(\because y = x - \frac{a}{k}\right)$$

$$\text{or, } t_1 = -\sqrt{\frac{a}{kg}}\left[\sin^{-1}\frac{ky}{a(nk-1)}\right]_{(nk-1)\frac{a}{k}}^{-\frac{a}{k}} = \sqrt{\frac{a}{kg}}\left[\sin^{-1}\frac{1}{nk-1} + \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{a}{kg}}\left[\pi - \cos^{-1}\frac{1}{nk-1}\right] (\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}).$$

***Example* 3.9.8.** A heavy particle of mass m is attached to one end of an elastic string of natural length a whose other end is fixed at O. The particle is let fall from rest at O. Show that part of the motion is simple harmonic, and that, if the greatest depth of the particle below O is $a\cot^2\frac{\theta}{2}$, the modulus of elasticity of the string is $\frac{1}{2}mg\tan^2\theta$, and the particle attains this depth in time $\sqrt{\frac{2a}{g}}\{1 + (\pi - \theta)\cot\theta\}$.

Solution : Let OA = a be the natural length of an elastic string of modulus of elasticity λ.

With fixed end at O, suppose a particle of mass attached to the end A is let fall from rest at O. Then the velocity acquired by the porticle on reaching the point A is $u = \sqrt{2ag}$ in vertically downward direction. The particle will then move downward under the tension of the string acting vertically upward and its weight mg acting vertically downward (See fig. 3.16).

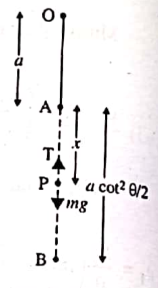

Fig. 3.16

Let at time t measured from the instant when the particle is at A, P let the position of the particle where AP = x. Tension acting at P is $T = \lambda\frac{x}{a}$. Hence the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - T \text{ or, } \frac{d^2x}{dt^2} = g - \frac{\lambda}{am}x \text{ or, } \frac{d^2y}{dt^2} = -\frac{\lambda}{am}y \quad \text{... (1)}$$

where $y = x - \dfrac{mga}{\lambda}$.

Equation (1) shows that the subsequent motion from A downwards is simple harmonic with centre of oscillation at $y = 0$ or $x = \dfrac{mga}{\lambda}$. Thus part of the motion of the particle is S.H.M.

To find the greatest depth of the particle below O, we find y where $\dfrac{dy}{dt} = 0$.

From(1), multiplying both sides by $2\dfrac{dy}{dt}$ and integrating, we get

$$\left(\frac{dy}{dt}\right)^2 = c_1 - \frac{\lambda}{am}y^2 \quad \text{... (2)}$$

At $t = 0$, $x = 0$, i.e., $y = -\dfrac{mga}{\lambda}$, $\dfrac{dx}{dt} = \dfrac{dy}{dt} = \sqrt{2ag}$.

Hence from (2), $c_1 = 2ag + \dfrac{am}{\lambda}g^2$.

Hence from (2),

$$\left(\frac{dy}{dt}\right)^2 = 2ag + \frac{am}{\lambda}g^2 - \frac{\lambda}{am}y^2 \quad \text{... (3)}$$

At the greatest depth below O, $\dfrac{dy}{dt} = 0$. So, from (3),

$$y^2 = \frac{am}{\lambda}\left(2ag + \frac{am}{\lambda}g^2\right) \quad \text{... (4)}$$

$$\text{or, } \left(x - \frac{mga}{\lambda}\right)^2 = \frac{2ma^2g}{\lambda} + \frac{a^2m^2}{\lambda^2}g^2$$

$$\text{or, } x^2 - \frac{2mga}{\lambda}x = \frac{2ma^2g}{\lambda} \quad \text{... (5)}$$

If the greatest depth below O is $a\cot^2\frac{1}{2}\theta$ (= OB)

then $x = a\cot^2\frac{1}{2}\theta - a$ (= AB)

So from (5),

$$\left(a\cot^2\tfrac{1}{2}\theta - a\right)^2 - \frac{2mga}{\lambda}\left(a\cot^2\tfrac{1}{2}\theta - a\right) = \frac{2m^2ag}{\lambda}$$

$$\Rightarrow \frac{2mga}{\lambda}\cdot a\cot^2\frac{\theta}{2} = a^2\left(\cot^2\frac{\theta}{2} - 1\right)^2 \Rightarrow \lambda = 2mg\left(\frac{\cot\frac{\theta}{2}}{\cot^2\frac{\theta}{2} - 1}\right)^2$$

$$= \frac{1}{2}mg\left(\frac{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\right)^2 = \frac{1}{2}mg\tan^2\theta.$$

If t_1 be the time from O to A then $t_1 = \dfrac{u}{g} = \sqrt{\dfrac{2a}{g}}$.

Let t_2 be the time from A to B. Then at $x = 0$, $t = 0$,

and at $x = AB = a\left(\cot^2\dfrac{\theta}{2} - 1\right)$, $t = t_2$.

From (3), we have, on taking $\lambda = \frac{1}{2}mg\tan^2\theta$

$$\left(\frac{dy}{dt}\right)^2 = 2ag + 2ag\cot^2\theta - \frac{g}{2a}\tan^2\theta\, y^2 = 2ag\,\text{cosec}^2\theta - \frac{g}{2a}\tan^2\theta\, y^2$$

$$= \frac{g}{2a}\tan^2\theta[4a^2\,\text{cosec}^2\theta\cot^2\theta - y^2]$$

$$\text{or, } \frac{dy}{dt} = \sqrt{\frac{g}{2a}}\tan\theta\sqrt{(2a\,\text{cosec}\,\theta\cot\theta)^2 - y^2}$$

$$\text{or, } \int_{t=0}^{t_2} dt = \sqrt{\frac{2a}{g}}\cot\theta\int_{y=-2a\cot^2\theta}^{2a\,\text{cosec}\,\theta\cot\theta}\frac{dy}{\sqrt{(2a\,\text{cosec}\,\theta\cot\theta)^2 - y^2}}$$

($\because$ when $x = 0$, $y = -\dfrac{mga}{\lambda} = -2a\cot^2\theta$ and when $x = a\left(\cot^2\dfrac{\theta}{2} - 1\right)$,

$y = a\left(\cot^2\frac{\theta}{2} - 1\right) - 2a\cot^2\theta$, which on simplification becomes $y = 2a$ cosecθ cotθ)

$$\text{or, } t_2 = \sqrt{\frac{2a}{g}}\cot\theta\left[\sin^{-1}\frac{y}{2a\operatorname{cosec}\theta\cot\theta}\right]_{y=-2a\cot^2\theta}^{y=2a\operatorname{cosec}\theta\cot\theta}$$

$$= \sqrt{\frac{2a}{g}}\cot\theta\cdot\left[\frac{\pi}{2} + \sin^{-1}\cos\theta\right] = \sqrt{\frac{2a}{g}}[\pi - \theta]\cot\theta.$$

Hence time to attain the depth $a\cot^2\frac{\theta}{2}$ below O is

$$t_1 + t_2 = \sqrt{\frac{2a}{g}}[1 + (\pi - \theta)\cot\theta].$$

Example 3.9.9. Two masses M and m are connected by a light elastic string of natural length a and modulus of elasticity λ, and the system is placed on a smooth horizontal table with the string perpendicular to an edge just unstretched, M lying on the table and m just having over. If the system be now allowed to move, prove that M will leave the table after a time t given by

$$Ma^2n^2 = \lambda g\left(\frac{1}{2}t^2 - \frac{2}{n^2}\sin^2\frac{nt}{2}\right), \quad \text{where } n^2 = \frac{\lambda}{a}\left(\frac{1}{M} + \frac{1}{m}\right).$$

Solution : Let AB$(= a)$ be the natural length of a light elastic string of modulus of elasticity λ.

The string AB is placed on a smooth horizontal table with the string perpendicular to an edge.

A mass M is attached at the end A and a mas m is attached at the end B, which is just hanging over.

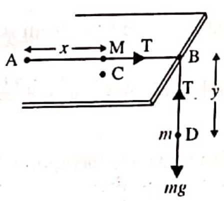

Fig. 3.17

If the system is allowed to move then suppose the mass m falls through a distance $y =$ BD and the mass M moves through a distance x (= AC) at time t (see fig. 3.17).

If T be the tension of the string at time t then $T = \lambda\frac{CD - AB}{AB} = \lambda\frac{y + a - x - a}{a}$

$$\text{or, } T = \lambda\frac{y - x}{a} \quad \text{... (1)}$$

Then equation of motion of the masses M and m are respectively

$$M\frac{d^2x}{dt^2} = T \quad \text{... (2)}$$

$$\text{and} \quad m\frac{d^2y}{dt^2} = mg - T \quad \text{... (3)}$$

Subtracting (2) from (3), we have, by using (1)

$$\frac{d^2(y-x)}{dt^2} = g - \frac{\lambda}{a}\left(\frac{1}{M} + \frac{1}{m}\right)(y - x)$$

$$\text{or, } \frac{d^2(y-x)}{dt^2} = g - n^2(y - x) \quad \text{or, } \frac{d^2z}{dt^2} = -n^2z \quad \text{... (4)}$$

where $z = y - x - \frac{g}{n^2}$ and $n^2 = \frac{\lambda}{a}\left(\frac{1}{M} + \frac{1}{m}\right)$. ... (5)

The general solution of the equation (4) is

$$z = c_1\cos nt + c_2\sin nt \quad \text{... (6)}$$

where c_1, c_2 are arbitrary constants to be determined from initial condition.

Initially, when $t = 0$, $x = 0$, $y = 0$ so that $z = -\frac{g}{n^2}$

and $\frac{dx}{dt} = 0 = \frac{dy}{dt}$, hence $\frac{dz}{dt} = 0$.

From (6) $\frac{dz}{dt} = n[-c_1\sin nt + c_2\cos nt]$.

Using initial condition, we get

$$-\frac{g}{n^2} = c_1 \text{ and } c_2 = 0.$$

Thus, from (6), we obtain

$$z = -\frac{g}{n^2}\cos nt \quad \text{or, } y - x = \frac{g}{n^2}(1 - \cos nt) \quad \text{... (7)}$$

From (2), $\frac{d^2x}{dt^2} = \frac{T}{M} = \frac{\lambda}{aM}(y - x)$

or, $\frac{d^2x}{dt^2} = \frac{\lambda g}{aMn^2}(1 - \cos nt)$ (by (7)).

Integrating, $\frac{dx}{dt} = c_2 + \frac{\lambda g}{aMn^2}\left(t - \frac{1}{n}\sin nt\right)$

At $t = 0$, $\frac{dx}{dt} = 0$. Hence $c_2 = 0$.

Therefore, $\frac{dx}{dt} = \frac{\lambda g}{aMn^2}\left(t - \frac{1}{n}\sin nt\right)$.

Integrating further, we have

$$x = c_3 + \frac{\lambda g}{aMn^2}\left(\frac{1}{2}t^2 + \frac{1}{n^2}\cos nt\right).$$

At $t = 0$, $x = 0$, $c_3 = -\frac{\lambda g}{aMn^4}$. Thus we have

$$x = \frac{\lambda g}{aMn^2}\left[\frac{1}{2}t^2 + \frac{1}{n^2}(\cos nt - 1)\right] \qquad \text{... (8)}$$

The time t taken by the mass M to pass over the table is obtained by putting $x = a$ in (8) and is given by

$$Ma^2n^2 = \lambda g\left(\frac{1}{2}t^2 - \frac{2}{n^2}\sin^2\frac{nt}{2}\right).$$

***Example* 3.9.10.** A heavy particle is attached to the lower end of an elastic string the upper end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and is then let go. Show that the particle will return to this point in time $\sqrt{\frac{a}{g}}\left(2\sqrt{3} + \frac{4\pi}{3}\right)$.

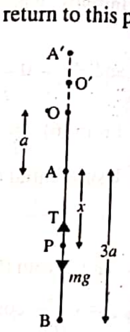

Fig. 3.18

***Solution*:** Let OA($= a$) be the natural length of an elastic string of modulus of elasticity $\lambda(= mg)$ with end O fixd and a heavy particle of mass m is attached to the lower end A. The end A is pulled vertially down to the point B such that OB = $4a$.

Then total extension of the string is AB = $3a$. The end B is then let go. Let at time t, P be the position of the particle such that AP = x. Then forces at P are tension T = $mg\frac{x}{a}$ in upward direction and the weight mg in downward direction (See fig. 3.18).

The equations of motion is

$$m\frac{d^2x}{dt^2} = mg - T \quad \text{or,} \quad \frac{d^2x}{dt^2} = g - \frac{g}{a}x \quad \text{or,} \quad \frac{d^2y}{dt^2} = -\frac{g}{a}y \qquad \text{... (1)}$$

where $y = x - a$.

Multiplying equation (1) by $2\frac{dy}{dt}$ and integrating, we get

$$\left(\frac{dy}{dt}\right)^2 = c_1 - \frac{g}{a}y^2 \qquad \text{... (2)}$$

where c_1 is constant of integration.

At $t = 0$ (measured from B), $x = 3a$, so $y = 2a$ and $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

Hence $c_1 = 4ag$. Thus from (2), we obtain

$$\left(\frac{dy}{dt}\right)^2 = \frac{g}{a}(4a^2 - y^2) \Rightarrow \frac{dy}{dt} = -\sqrt{\frac{g}{a}}\sqrt{4a^2 - y^2} \qquad \text{... (3)}$$

negative sign is taken, since particle is moving in the direction of y decreasing.

If t_1 be the time from B to A i.e. from $y = 2a$ to $y = -a$ then from (3),

$$\int_0^{t_1} dt = \sqrt{\frac{a}{g}}\int_{y=2a}^{-a}\frac{-dy}{\sqrt{4a^2 - y^2}} = \sqrt{\frac{a}{g}}\left[\sin^{-1}\frac{y}{2a}\right]_{y=-a}^{2a} = \sqrt{\frac{a}{g}}\left[\frac{\pi}{2} + \frac{\pi}{6}\right]$$

$$\Rightarrow t_1 = \sqrt{\frac{a}{g}}\cdot\frac{2\pi}{3}.$$

From (3), the velocity with which the particle will reach the point A is obtained by taking $y = -a$ and is given by $v_1 = -\sqrt{\frac{g}{a}}\cdot\sqrt{3}\,a$, negative sign indicates that the particle will go beyond A when the string will become slack and so no tension exists.

The particle will move upto the point O′ against gravity such that OO′ $= \frac{v_1^2}{2g} = \frac{3}{2}a$.

Since, OO′ $< 2a(=$ AA′), so before the string regains its natural length in its upward motion velocity of the particle vanishes and hence the particle will then fall freely, from rest at O′, under gravity upto the point A after which tension will begin to act again and the particle will descend the height O′A with velocity at A equal to $\sqrt{2\cdot\frac{3a}{2}g} = \sqrt{3ag}$ in downward direction, and this velocity is same as v_1 in magnitude in the case of upward motion of the particle.

Thus the particle will reach the point B again and this downward motion will be represented by the same equation (1).

The time t_2 for the upward motion from A to O′ is given by $t_2 = \frac{\sqrt{3ag}}{g} = \sqrt{\frac{3a}{g}}$.

From symmetry, the time of a complete oscillation, from B to O′ and back to B is

$$2(t_1 + t_2) = 2\left[\sqrt{\frac{3a}{g}} + \sqrt{\frac{a}{g}}\cdot\frac{2\pi}{3}\right] = \sqrt{\frac{a}{g}}\left[2\sqrt{3} + \frac{4\pi}{3}\right].$$

***Example* 3.9.11.** A particle possesses simultaneously two S.H.M.'s of the same period along the same straight line about a common centre, their amplitude being a and $a\sqrt{3}$, and the phase of the latter motion being a quarter of a period in advance of the former. Show that the resultant motion is simple harmonic of amplitude $2a$, and phase one-sixth of a period in advance of the first.

Solution : Let $T = \frac{2\pi}{\sqrt{\mu}}$ be the common time period of two S.H.M.'s along the same straight line about a common centre.

Then the displacement-time relation describing the two S.H.M.'s are

$$x_1 = a\cos(\sqrt{\mu}\,t + \varepsilon_1) \quad \text{... (1)}$$

and

$$x_2 = a\sqrt{3}\cos(\sqrt{\mu}\,t + \varepsilon_2) \quad \text{... (2)}$$

The phase of the first S.H.M is $t + \frac{\varepsilon_1}{\sqrt{\mu}}$ and that of the second S.H.M. is $t + \frac{\varepsilon_2}{\sqrt{\mu}}$. According to the problem,

$$t + \frac{\varepsilon_2}{\sqrt{\mu}} = t + \frac{\varepsilon_1}{\sqrt{\mu}} + \frac{1}{4}T \Rightarrow \varepsilon_2 - \varepsilon_1 = \frac{\pi}{2} \quad \text{... (3)}$$

From (1) and (2), on addition, we have

$$x = x_1 + x_2 = a[\cos\sqrt{\mu}\,t(\cos\varepsilon_1 + \sqrt{3}\cos\varepsilon_2) - \sin\sqrt{\mu}\,t(\sin\varepsilon_1 + \sqrt{3}\sin\varepsilon_2)]$$

$$= A\cos(\sqrt{\mu}\,t + \varepsilon) \quad \text{... (4)}$$

where

$$\left.\begin{aligned} A\cos\varepsilon &= a\cos\varepsilon_1 + \sqrt{3}\,a\cos\varepsilon_2 \\ \text{and}\quad A\sin\varepsilon &= a\sin\varepsilon_1 + \sqrt{3}\,a\sin\varepsilon_2 \end{aligned}\right\} \quad \text{... (5)}$$

From (5), we have

$$A = \sqrt{\left[a^2 + 3a^2 + 2\sqrt{3}a^2\cos(\varepsilon_2 - \varepsilon_1)\right]} = 2a \quad \text{(by (3))}$$

and
$$\tan\varepsilon = \frac{\sin\varepsilon_1 + \sqrt{3}\sin\varepsilon_2}{\cos\varepsilon_1 + \sqrt{3}\cos\varepsilon_2} = \frac{\sin\varepsilon_1 + \sqrt{3}\cos\varepsilon_1}{\cos\varepsilon_1 - \sqrt{3}\sin\varepsilon_1} = \frac{\sqrt{3} + \tan\varepsilon_1}{1 - \sqrt{3}\tan\varepsilon_1} = \tan\left(\frac{\pi}{3} + \varepsilon_1\right)$$

$$\Rightarrow \varepsilon = \frac{\pi}{3} + \varepsilon_1$$

and hence
$$t + \frac{\varepsilon}{\sqrt{\mu}} = t + \frac{\varepsilon_1}{\sqrt{\mu}} + \frac{\pi}{3\sqrt{\mu}} \Rightarrow \left(t + \frac{\varepsilon}{\sqrt{\mu}}\right) - \left(t + \frac{\varepsilon_1}{\sqrt{\mu}}\right) = \frac{1}{6}T.$$

Thus from (4), resultant of two S.H.M.'s is a S.H.M of amplitude $2a$ and its phase is $\frac{1}{6}$th of a period in advance of the phase of the first S.H.M.

***Example* 3.9.12.** A particle is executing a simple harmonic oscillation of amplitude a under an attraction $\frac{\mu x}{a}$. If a small disturbing force $\frac{\lambda x^3}{a^3}$ towards the centre be introduced (the amplitude being unchanged) show that the period is, to a first approximation, decreased in the ratio $\left(1 - \frac{3\lambda}{8\mu}\right) : 1$.

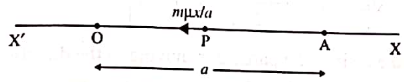

Fig. 3.19(a)

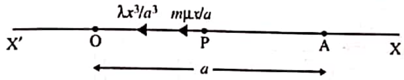

Fig. 3.19(b)

Solution : In fig. 3.19(a), we have a case of free oscillation of a particle of mass m under the attraction $\frac{m\mu x}{a}$ towards the centre of oscillation O, x being distance of the particle at P from O at time t.

Then equation of motion is $\quad m\frac{d^2x}{dt^2} = -m\frac{\mu}{a}x.$

Hence the time period is $\quad T_1 = 2\pi\sqrt{\frac{a}{\mu}} \quad \text{... (1)}$

In fig. 3.19(b), we have a case of oscillation under attraction $\frac{m\mu x}{a}$ towards O, together with a small disturbing force $\frac{m\lambda x^3}{a^3}$ towards O, the amplitude being unchanged.

Then the equation of motion is

$$m\frac{d^2x}{dt^2} = -m\frac{\mu}{a}x - m\frac{\lambda}{a^3}x^3 \quad \text{... (2)}$$

Multiplying equation (2) by $2\frac{dx}{dt}$ and integrating we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{\mu}{a}x^2 - \frac{\lambda}{2a^3}x^4 + c_1 \quad \text{... (3)}$$

where c_1 is a constant of integration.

Since 'a' is the amplitude of this motion also, so $\frac{dx}{dt} = 0$ where $x = a$. Therefore, $c_1 = \mu a + \frac{\lambda}{2}a$. Then from (3),

$$\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{a}(a^2 - x^2) + \frac{\lambda}{2a^3}(a^4 - x^4) = \frac{\mu}{a}(a^2 - x^2).\left[1 + \frac{\lambda}{2\mu a^2}(a^2 + x^2)\right]$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{\frac{\mu}{a}}\sqrt{a^2 - x^2}\left[1 + \frac{\lambda}{2\mu a^2}(a^2 + x^2)\right]^{\frac{1}{2}} \quad \text{...(4)}$$

negative sign is taken, since the particle is moving in the direction of x decreasing.

If t_1 be the time required from its position instantaneous rest at A to the centre of force O, then we have from (4),

$$\int_{t=0}^{t_1} dt = -\sqrt{\frac{a}{\mu}}\int_{x=a}^{0}\frac{dx}{\sqrt{a^2 - x^2}\left[1 + \frac{\lambda}{2\mu a^2}\left(a^2 + x^2\right)\right]^{\frac{1}{2}}}$$

$$= \sqrt{\frac{a}{\mu}}\int_{x=0}^{a}\frac{\left[1 - \frac{\lambda}{4\mu a^2}\left(a^2 + x^2\right)\right]dx}{\sqrt{a^2 - x^2}}$$

($\because \lambda$ is small, we retain only linear term and neglect all other higher powers of λ)

$$= \sqrt{\frac{a}{\mu}}\left\{\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} + \frac{\lambda}{4\mu a^2}\int_0^a \frac{a^2 - x^2 - 2a^2}{\sqrt{a^2 - x^2}}dx\right\}$$

$$= \sqrt{\frac{a}{\mu}}\left\{\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} + \frac{\lambda}{4\mu a^2}\int_0^a \sqrt{a^2 - x^2}\,dx - \frac{\lambda}{2\mu}\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}\right\}$$

$$= \sqrt{\frac{a}{\mu}}\left\{\left(1 - \frac{\lambda}{2\mu}\right)\left[\sin^{-}\frac{x}{a}\right]_0^a + \frac{\lambda}{4\mu a^2}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a\right\}$$

$$= \sqrt{\frac{a}{\mu}}\left\{\left(1 - \frac{\lambda}{2\mu}\right)\frac{\pi}{2} + \frac{\lambda}{8\mu}.\frac{\pi}{2}\right\} = \sqrt{\frac{a}{\mu}}\frac{\pi}{2}\left(1 - \frac{3\lambda}{8\mu}\right).$$

Hence, from symmetry, time-period of the second oscillation is

$$T_2 = 4t_1 = 2\pi\sqrt{\frac{a}{\mu}}\left(1 - \frac{3\lambda}{8\mu}\right).$$

Thus $T_2 : T_1 = \left(1 - \frac{3\lambda}{8\mu}\right) : 1$, to a first approximation.