Dept. of Mathematics

2^{nd} Semester

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Sets in **R**

Topics : Neighbourhood, Interior point, Open set

Neighbourhood

Let $c \in \mathbb{R}$. A subset $S \subseteq \mathbb{R}$ is said to be a neighbourhood of c if there exists an open interval (a, b) such that $c \in (a, b) \subseteq S$.

Note:

- An open bounded interval containing the point c is a neighbourhood of c. Such a neighbourhood of c is denoted by $N(c)$.
- A closed bounded interval containing the point c may not be a neighbourhood of c. For example, $1 \in [1, 3]$ but $[1, 3]$ is not a neighbourhood of 1.
- Let $c \in \mathbb{R}$ and $\delta > 0$, the open interval $(c-\delta, c+\delta)$ is said to the δ-neighbourhood of c and is denoted by $N(c, \delta)$. clearly, the δ-neighbourhood of c is an open interval symmetric about c.

Theorem

Let $c \in \mathbb{R}$. The union of two neighbourhood of c is a neighbourhood of c.

Proof

Let $S_1 \subseteq \mathbb{R}$, $S_2 \subseteq \mathbb{R}$ be two neighbourhood of c.

$\Rightarrow$ there exist open intervals (a_1, b_1) (a_2, b_2) s.t

$c \in (a_1, b_1) \subseteq S_1$ and $c \in (a_2, b_2) \subseteq S_2$

$\Rightarrow c \in (a_1, b_1) \subseteq S_1 \cup S_2$

$\Rightarrow S_1 \cup S_2$ is a neighbourhood of c

Note:

The arbitrary union of neighbourhoods of c is a neighbourhood of c. [check]

Theorem

Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c.

proof

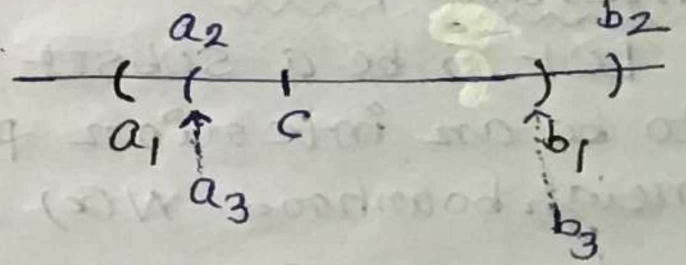
let $S_1 \subseteq \mathbb{R}$, $S_2 \subseteq \mathbb{R}$ be two neighbourhoods of c.

$\Rightarrow$ there exist open intervals (a_1, b_1), (a_2, b_2) s.t

$c \in (a_1, b_1) \subseteq S_1$ and $c \in (a_2, b_2) \subseteq S_2$

let $a_3 = \max\{a_1, a_2\}$

$b_3 = \min\{b_1, b_2\}$



$\Rightarrow (a_3, b_3) = (a_1, b_1) \cap (a_2, b_2)$

and $c \in (a_3, b_3)$

$\Rightarrow (a_3, b_3) \subseteq (a_1, b_1)$ and $(a_3, b_3) \subseteq (a_2, b_2)$

$\Rightarrow (a_3, b_3) \subseteq S_1$ and $(a_3, b_3) \subseteq S_2$

$\Rightarrow (a_3, b_3) \subseteq S_1 \cap S_2$

$\Rightarrow c \in (a_3, b_3) \subseteq S_1 \cap S_2$

$\Rightarrow S_1 \cap S_2$ is a neighbourhood of c.

Note:

- The intersection of a finite number of neighbourhoods of a point c is a neighbourhood of c. [check]

- The intersection of an infinite number of neighbourhoods of a point c may not be a neighbourhood of c.

For example, for every $n \in \mathbb{N}$, $(-\frac{1}{n}, \frac{1}{n})$ is a neighbourhood of 0.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$ [check]

This is not a neighbourhood of 0.

Interior point

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an interior point of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subseteq S$.

The set of all interior points of S is said to be the interior of S and is denoted by $\text{int}\, S$ (or by S^0).

Note:

From definition it follows that $S^0 \subseteq S$ for any set $S \subseteq \mathbb{R}$.

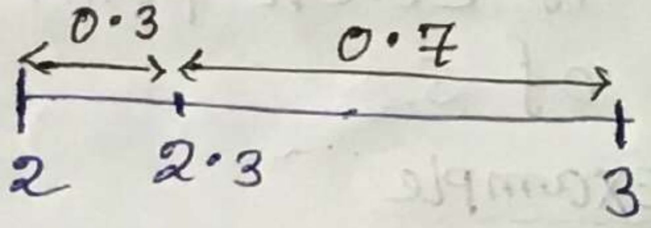
Example

- To show $2 \cdot 3$ is an interior point of $[2, 3]$

Solⁿ

Aim:

$$\boxed{\exists\ \delta > 0 \text{ such that } N(2\cdot 3, \delta) \subseteq [2, 3]}$$



choose $\delta = \min\{2\cdot 3 - 2,\ 3 - 2\cdot 3\} = \min\{0\cdot 3,\ 0\cdot 7\}$

$= 0\cdot 3 > 0.$

then $N(2\cdot 3, \delta) = N(2\cdot 3, 0\cdot 3) = (2\cdot 3 - 0\cdot 3,\ 2\cdot 3 + 0\cdot 3)$

$= (2, 2\cdot 6) \subseteq [2, 3]$

$\Rightarrow$ $2\cdot 3$ is an interior point of $[2, 3]$

- Prove that $1\cdot5$ is not an interior point of $[1\cdot5, 2]$

Sol^n

Aim:

$$\boxed{\forall \delta > 0 \quad N(1\cdot5, \delta) \not\subseteq [1\cdot5, 2]}$$

Take any $\delta > 0$.

To show $N(1\cdot5, \delta) \not\subseteq [1\cdot5, 2]$

i.e $\exists\, y \in N(1\cdot5, \delta)$ but $y \notin [1\cdot5, 2]$

choose $y = 1\cdot5 - \delta/2 < 1\cdot5$

$\Rightarrow y \in N(1\cdot5, \delta)$ but $y \notin [1\cdot5, 2]$

$\Rightarrow 1\cdot5$ is not an interior point of $[1\cdot5, 2]$

Open set

Let $S \subseteq \mathbb{R}$. S is said to be an open set if each point of S is an interior point of S.

Example

- To show $(1,2)$ is open in $\mathbb{R}$.

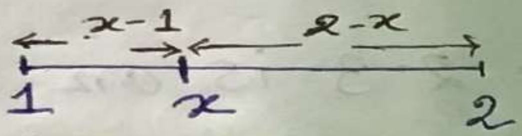
Sol^n

Aim:

$$\boxed{\forall x \in (1,2)\ \exists\ \delta_x > 0 \text{ s.t } N(x,\delta) \subseteq (1,2)}$$

Take any $x \in (1,2)$

choose $\delta_x = \min\{x-1, 2-x\} > 0$



To show $N(x,\delta_x) \subseteq (1,2)$

i.e $\forall y \in N(x,\delta_x) \Rightarrow y \in (1,2)$

Take any $y \in N(x,\delta_x)$

$\Rightarrow y > x - \delta_x \geqslant x-(x-1) = 1$ [Since $\delta_x \leq (x-1)$]

Also $y < x + \delta_x \leq x + (2-x) = 2$ [Since $\delta_x \leq (2-x)$]

$\Rightarrow y \in (1,2)$

Hence $N(x,\delta_x) \subseteq (1,2)$

- Show that $(1, 2]$ is not open in $\mathbb{R}$

Sol^n

$\boxed{\text{Aim:}}$

$\boxed{\exists\, x \in (1, 2] \text{ s.t } \forall\, \delta > 0 \quad N(x, \delta) \not\subseteq (1, 2]}$

choose $x = 2$ and take any $\delta > 0$.

To show $N(2, \delta) \not\subseteq (1, 2]$:

i.e $\exists\, y \in N(2, \delta)$ s.t $y \notin (1, 2]$

choose $y = 2 + \delta/2$

$\Rightarrow y \in N(2, \delta)$ but $y \notin (1, 2]$ [Since $y > 2$]

This is true for any $\delta > 0$

$\Rightarrow N(x, \delta) \not\subseteq (1, 2] \quad \forall\, \delta > 0.$

- Show that the set $S = \{x \in \mathbb{R} : |x-1| + |x-2| < 3\}$ is an open set.

Sol^n

Let $x \in S$

- If $x < 1$ then

$\Rightarrow |x-1| + |x-2| = (1-x) + (2-x) = 3 - 2x < 3$

$\Rightarrow x > 0$

- If $x \geq 2$

$\Rightarrow |x-1| + |x-2| = (x-1) + (x-2) = 2x - 3 < 3$

$\Rightarrow x < 3$

• If $1 \le x < 2$

$\Rightarrow |x-1| + |x-2| = (x-1) + (2-x) = 1 < 3$

which is always true.

So, $S = (0, 3)$

Now, $(0, 3)$ is open because $\forall x \in (0, 3)$

we can take $\delta_x = \min\{x, 3-x\}$ such that

$N(x, \delta_x) \subseteq (0, 3)$.

Theorem

The union of any arbitrary collection of open sets of real numbers is an open set.

Proof

Let $\{A_i : i \in I\}$ be any collection of open sets in $\mathbb{R}$, where I is any arbitrary index set.

Now we show that $A = \bigcup_{i \in I} A_i$ is open.

Aim:

$\forall\, a \in A \ \exists\, \delta_a > 0$ s.t $N(a, \delta_a) \subseteq A$

Take any $a \in A$ so $\exists\, \lambda \in I$

S.t $a \in A_\lambda$.

Since A_λ is an open set

$\Rightarrow \exists\ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq A_\lambda$

$\Rightarrow N(a, \delta_a) \subseteq A_\lambda \subseteq A$

$\Rightarrow$ a is an interior point of A

since a is an arbitrary point of A

$\Rightarrow A$ is an open set.

Theorem

The intersection of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof

Let $A_1, A_2, \ldots, A_m$ be m open sets in $\mathbb{R}$

Let $A = A_1 \cap A_2 \cap \cdots \cap A_m$

To show A is an open set.

Aim:

$\forall a \in A \ \exists \ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq A$

Case I : $A = \phi$ (empty set)

$\Rightarrow A$ is open, since ϕ is an open set (check)

Case II : $A \neq \phi$

Aim:

$\forall a \in A \ \exists \ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq A$

let $a \in A$

$\Rightarrow a \in A_i$ for each $i = 1, 2, \ldots, m$

But since each A_i is an open set

$\Rightarrow \exists\ \delta_a^i > 0$ s.t. $N(a, \delta_a^i) \subseteq A_i$ for $i = 1, 2, \ldots, m$

choose $\delta_a = \min\{\delta_a^1, \delta_a^2, \ldots \delta_a^m\}$

$\Rightarrow \delta_a > 0$ and $N(a, \delta_a) \subseteq N(a, \delta_a^i)\ \forall i = 1, 2, \ldots, m$

$\Rightarrow N(a, \delta_a) \subseteq A_i\ \forall i = 1, 2, \ldots, m$

$\Rightarrow N(a, \delta_a) \subseteq A_1 \cap A_2 \cap \cdots \cap A_m = A$

$\Rightarrow a$ is an interior point of A

Since a is an arbitrary point of A

$\Rightarrow A$ is an open set.

Note:

The intersection of an infinite collection of open sets need not always be an open set.

choose $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \quad n \in \mathbb{N}$

Here each A_n is an open set.

But $\bigcap_{n=1}^{\infty} A_n = \{0\}$ is not an open set.

Theorem

Let $S \subseteq \mathbb{R}$. Then int S (S°) is the largest open set contained in S.

Proof

Step 1

To show S° is open

Aim:

$\forall a \in S^\circ \ \exists \ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq S^\circ$

Case I : If $S^\circ = \phi$, then vacuously true.

Case II : If $S^\circ \neq \phi$

Take any $a \in S^\circ$

$\Rightarrow \exists \ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq S$ [Since a is an interior point of S]

Aim:

$N(a, \delta_a) \subseteq S^\circ$ i.e. $\forall x \in N(a, \delta_a) \Rightarrow x \in S^\circ$

$\Rightarrow a$ is an interior point of S°.

Take any $x \in N(a, \delta_a)$

Since $N(a, \delta_a)$ is a neighbourhood of x

and $N(a, \delta_a) \subseteq S$

$\Rightarrow x$ is an interior point of S i.e. $x \in S^\circ$

i.e. $x \in N(a, \delta_a) \Rightarrow x \in S^\circ$

$\Rightarrow N(a, \delta_a) \subseteq S^\circ$

$\Rightarrow a$ is an interior point of S°

$\Rightarrow S^\circ$ is an open set.

Step 2

For any $A \subseteq S$ if A is open in $\mathbb{R}$

$\Rightarrow A \subseteq S^0$

Aim:

$\forall a \in A \Rightarrow a \in S^0$

Take any $a \in A$

Since A is open

$\Rightarrow \exists\ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq A$

Again $A \subseteq S$

$\Rightarrow N(a, \delta_a) \subseteq A \subseteq S.$

$\Rightarrow a \in S^0$

$\Rightarrow A \subseteq S^0$

Theorem

Every nonempty open set can be expressed as a union of open intervals

proof

Let $A \neq \Phi$ be an open subset of $\mathbb{R}$

$\Rightarrow \forall a \in A \ \exists \ \delta_a > 0$ s.t $N(a, \delta_a) \subseteq A$

$$\boxed{\text{Aim: } A = \bigcup_{a \in A} N(a, \delta_a)}$$

Take $B = \bigcup_{a \in A} N(a, \delta_a)$, then $\forall a \in A$, $N(a, \delta_a) \subseteq A \Rightarrow B \subseteq A$

Also $\forall a \in A$, $a \in N(a, \delta_a) \subseteq B \Rightarrow A \subseteq B$

$\therefore A = B$

Examples

- Let $S = (0\ 1]$ and $T = \{\frac{1}{n} : n = 1, 2, 3, \dots\}$. show that $S \setminus T$ is an open set.

Sol^n

$$S \setminus T = (\tfrac{1}{2}\ 1) \cup (\tfrac{1}{3}, \tfrac{1}{2}) \cup (\tfrac{1}{4}, \tfrac{1}{3}) \cup \cdots$$

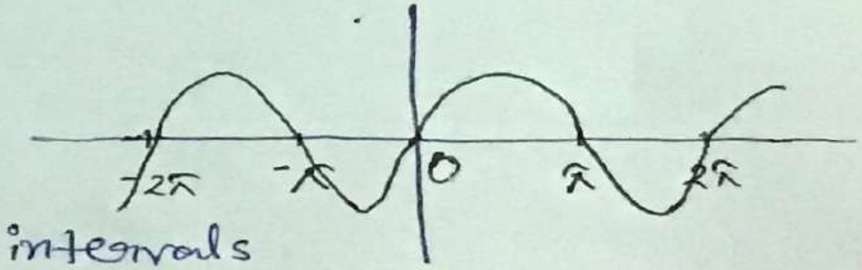
$\Rightarrow$ $S \setminus T$ is the union of an infinite number of open intervals

$\Rightarrow$ $S \setminus T$ is open [Since an open interval is an open set (check)]

- Let $S = \{x \in \mathbb{R} : \sin x \neq 0\}$. show that S is an open set.

Sol^n

$$S = \bigcup_{n \in \mathbb{Z}} ((n-1)\pi, n\pi)$$



$\Rightarrow$ S is the union of an infinite number of open intervals

$\Rightarrow$ S is an open set

Practice problems

1) Prove that the set S is an open set, where

(i) $S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 < 0\}$

(ii) $S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 > 0\}$

(iii) $S = A \setminus B$ where $A = (0, 1)$, $B = \{\frac{1}{2^n} : n \in \mathbb{N}\}$

2) Let G be an open set in $\mathbb{R}$ and S be a non-empty finite subset of G. Prove that $G \setminus S$ is an open set.

3) Prove that an open interval is an open set.