Course Material on Magnetostatics

Lorentz force

Force due to a magnetic field $\vec{B}(\vec{r})$ on a charged particle (Q) traveling with velocity $\vec{v}$ is given by :

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B}(\vec{r})) \tag{1}$$

This obviously means that the force is always perpendicular to the direction of motion. Hence this force does no work :

$$dW = \vec{F}.\vec{dl} = Q(\vec{v} \times \vec{B}).\vec{dl} = Q(\vec{v} \times \vec{B}).\vec{v}dt = 0 \tag{2}$$

This means that the kinetic energy of the particle does not change. Thus the $|\vec{v}|$ does not change. Thus under this force only the direction of the particle changes. As an example if $\vec{v} = v\hat{i}$ and $\vec{B} = B\hat{j}$ then $\vec{v} \times \vec{B} = vB\hat{k}$. This also means that in a constant magnetic field the particle will under go motion in a circle in the plane orthogonal to the direction of magnetic field. Thus the Centripetal force is provided y the magnetic force. Hence:

$$\frac{mv^2}{R} = QvB \tag{3}$$

The radius of the circle is then given by :

$$R = \frac{mv}{QB} \tag{4}$$

The time period of this circular motion is then given by:

$$\frac{2\pi R}{v} = 2\pi \frac{m}{QB} \tag{5}$$

Thus the trajectory of the particle in this case is given by :

$$\vec{r}(t) = R(\cos(\frac{QB}{m}t + \phi)\hat{i} + \sin(\frac{QB}{m}t + \phi)\hat{k}) \tag{6}$$

As a generalization let us consider the case where $\vec{v} = v_{perp}\hat{i} + v_{para}\hat{j}$, where we have the component of velocity v_{para} which is parallel to $\vec{B}$. There is no force due to this component. The trajectory of the particle due to this is then given by :

$$\vec{r}(t) = v_{para}t\hat{j} + \frac{mv_{perp}}{QB}(\cos(\frac{QB}{m}t + \phi)\hat{i} + \sin(\frac{QB}{m}t + \phi)\hat{k}) \tag{7}$$

The above cyclic motion in a plane is called *cyclotron* motion while the trajectory in the later case is that of a Helix. Let us look at the general case where a charged particle is both under an electric and magnetic field. The force in that case is the Lorentz force:

$$\vec{F}_{Lorentz} = Q(\vec{E}(\vec{r}) + \vec{v} \times \vec{B}(\vec{r})) \tag{8}$$

1. Given a constant electric field $E\hat{k}$ and a constant magnetic field $B\hat{i}$, find the trajectory of a charged particle which is at rest at origin at time $t = 0$.

Currents

For dQ charge flowing through a point P in time dt the total current flowing through the point is given by :

$$I = \frac{dQ}{dt} \tag{9}$$

Now let us assume that charge is smeared on a one dimensional wire, and dl length carries dQ charge. This charge moves through point P in time dt. Hence this charge segment has velocity v, in direction of the flow. Thus the magnitude of the current can be written as :

$$I = \frac{dQ}{dt} = \frac{dQ}{dl}\frac{dl}{dt}, \quad \lambda = \frac{dQ}{dl} \tag{10}$$

Using this the force on the segment of the wire can be written as:

$$\begin{aligned} \vec{F} &= \int (\vec{v} \times \vec{B}) dQ \\ &= \int (\vec{v} \times \vec{B}) \frac{dQ}{dl} dl \\ &= \int \lambda \vec{v} \times \vec{B}) \frac{dQ}{dl} dl \\ &= \int (\vec{I} \times \vec{B}) dl \end{aligned} \tag{11}$$

Now the charge can also be smeared on a two dimensional surface as well. From which we can define surface current density :

$$\vec{K} = \frac{dQ}{da}\frac{d\vec{l}}{dt} = \sigma \vec{v}, \quad \sigma = \frac{dQ}{da} \tag{12}$$

while if the charge is smeared over a 3 dimensional region and the current flows through a 3 dimensional wire, we define volume current density:

$$\vec{J} = \frac{dQ}{d\tau}\frac{d\vec{l}}{dt} = \rho \vec{v}, \quad \rho = \frac{dQ}{d\tau} \tag{13}$$

In these two cases, the force on a corresponding current carrying element (area and volume respectively) is given by :

$$\vec{F} = \int (\vec{K} \times \vec{B}) da \quad \text{and} \quad \vec{F} = \int (\vec{J} \times \vec{B}) d\tau \tag{14}$$

For example if we have a uniform current I flowing through a cylinder of radius a, then the volume current density is given by

$$\vec{J} = \frac{I}{\pi a^2}\hat{z} \tag{15}$$

here $\hat{z}$ is the direction vector in direction of the axis of the cylinder. Note that this means that the total current flowing through the cylinder can be written as:

$$I = \vec{J}.\hat{z}\pi a^2 = \int_0^a \int_0^{2\pi} \rho \, d\rho \, d\phi \, \hat{z}.\vec{J} = \int d\vec{a}.\vec{J} \tag{16}$$

here $d\vec{a} = \rho d\rho d\phi \hat{z}$. Using this one can compute the total current flowing through a closed surface S :

$$I_{Total} = \oint_S \vec{J}.d\vec{a} \tag{17}$$

where the integral denotes over the closed surface. Using Gauss's divergence theorem we can write :

$$\oint_S \vec{J}.d\vec{a} = \int_V \vec{\nabla}.\vec{J} \, d\tau \tag{18}$$

Now since charge is conserved this must be negative of the rate of change of total charge included in the volume enclosed, i.e. :

$$I_{Total} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho d\tau = -\int_V \frac{\partial \rho}{\partial t} d\tau \tag{19}$$

where we have taken the derivative inside the integral, and since the integration variables are dummy i.e. $d\tau = dx'dy'dz'$ and the volume charge density $\rho(x', y', z', t)$ is a function of these variables for a fixed coordinate system. Hence:

$$\int_V \vec{\nabla}.\vec{J} \, d\tau = -\int_V \frac{\partial \rho}{\partial t} d\tau \tag{20}$$

But this being true for any arbitrary volume taken, we must have the integrands same. Thus:

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t} \tag{21}$$

Steady state is defined where there is no accumulation/depletion of charge in a given volume element, i.e. the ρ does not explicitly change with time. Thus $\vec{\nabla}.\vec{J} = 0$.

Biot-Savart Law

This gives us a formula to calculate the magnetic field at a point $\vec{r}$ for a given current distribution :

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{l} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} \tag{22}$$

where $\vec{\mathfrak{r}}$ is the displacement vector from the element $d\vec{l}$ to the point $\vec{r}$ where the magnetic field is to be calculated. The element $d\vec{l}$ traverses along the path via which the steady current

I flows. Hence the information about the geometry of the current configuration is encoded in this integral as well.
2. Find the magnetic field at a point, with shortest distance s from an infinite wire carrying current I.
3. Find the magnetic force per unit length on two parallel wires carrying current I_1 and I_2 and distance s apart from each other.
4. Show that the magnetic field at a point on the axis of a circular loop (radius R) and at height z from the plane of the loop is given by :

$$\frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z} \tag{23}$$

5.We can look at the case of a solenoid, which is a cylinder with wires wound around it. Let the windings per unit length be n and let the current flowing through each wire be I. This means the total number of coils in the interval dz along the axis of the cylinder is ndz and thus total current flowing through these coils is $= Indz$. Thus one can imagine the solenoid as a superposition of several circular loops each carrying current $Indz$. Then if the radius of the solenoid is a, each of these segments contribute a magnitude to the magnetic field given by :

$$dB = \frac{\mu_0 n I dz}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \tag{24}$$

making the change of variables $\tan\theta = \frac{a}{z}$, where θ is the angle subtended at the point on the axis of the cylinder where the magnetic field is measured, with respect to the an element of the elementary circular loop considered. Thus :

$$dz = -a\text{cosec}^2\theta \, d\theta, \quad \frac{1}{(z^2 + a^2)^{3/2}} = \frac{\sin^3\theta}{a^3} \tag{25}$$

Plugging this in and integrating, we have :

$$\begin{aligned} B &= \int_{\theta_1}^{\theta_2} \frac{a^2 \sin^3\theta}{a^3 \sin^2\theta} (-a d\theta) \\ &= -\frac{\mu_0 n I}{2} \int_{\theta_1}^{\theta_2} \sin\theta d\theta = \frac{\mu_0 n I}{2} \cos\theta \Big|_{\theta_1}^{\theta_2} \end{aligned} \tag{26}$$

If we have an infinite solenoid then $\theta_2 = 0$ and $\theta_1 = \pi$, and hence we obtain :

$$B = \mu_0 n I \tag{27}$$

the direction is along the axis of the cylinder i.e. in $\hat{z}$ direction.

Ampere's Law

If we look at the magnetic field due to an infinite long wire, and calculate the line integral of it around a circular loop, then we have:

$$\oint \vec{B}.d\vec{l} = \int \frac{\mu_0 I}{2\pi s}\hat{\phi}.d\vec{l} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi s}\hat{\phi}.s\,d\phi\hat{\phi} = \mu_0 I \tag{28}$$

From this we obtain the Ampere's law, which is valid for other current configurations as well, which states that the closed loop integral of the magnetic field is equal to μ_0 times the total current passing through the loop .

$$\oint \vec{B}.d\vec{l} = \mu_0 I_{enclosed} \tag{29}$$

We can go further and write this in differential form using Stokes theorem :

$$\oint \vec{B}.d\vec{l} = \int_S (\vec{\nabla} \times \vec{B}).d\vec{a} = \mu_0 I_{enclosed} = \mu_0 \int_S \vec{J}.d\vec{a} \tag{30}$$

since this is true for any arbitrary area S taken, we must have the integrands to be same :

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{31}$$

which is th differential form of the Ampere's law. As an analogy to electrostatics we observe that in that case we have Coloumb's law and Gauss's divergence theorem, and here we have Biot-Savart's law and Ampere's law.
6. Find the magnetic field due to suface current flowing in the $x - y$ plane and given by $\vec{K} = K\hat{i}$, using Ampere's law.
7. Find the magnetic due to a cylindrical wire of radius a for total current I flowing through it, if
a) Current is flowing on the surface of the cylinder.
b) A volume current density flows in the cylinder $\vec{J} = k\,s\hat{z}$, where s is the distance from the axis of the cylindrical wire.

Divergence and Curl of $\vec{B}$

We calculate the Divergence here directly from Biot-Savart's law. The most generic form of the magnetic field in terms of the volume current density is given by:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} dx'\,dy'\,dz' \tag{32}$$

where $\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$. Directly applying the divergence operator, but noting that it is with respect to the coordinates x, y, z and not with respect to the coordinates at which the source is i.e. x', y', z'. Thus :

$$\vec{\nabla}.\vec{B} = \frac{\mu_0}{4\pi}\int_V \vec{\nabla}.\left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) dx'\,dy'\,dz' \tag{33}$$

Using the formula for the divergence on curl of two functions :

$$\left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)$$
$$= \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}.(\vec{\nabla} \times \vec{J}(\vec{r'})) - \vec{J}(\vec{r'}).(\vec{\nabla} \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}) \tag{34}$$

The first term is zero since $\vec{J}$ is a function of $\vec{r'}$ here while we see that :

$$\vec{\nabla} \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = -\vec{\nabla} \times \vec{\nabla}|\vec{\mathfrak{r}}|^{-1} \tag{35}$$

Hence Curl of gradient of scalar. This is also zero , assuming commutativity of partial derivatives. Thus we find that :

$$\vec{\nabla}.\vec{B} = 0 \tag{36}$$

Any where in space. As a consequence of this we can use Gauss's divergence theorem on this to obtain that there is no- magnetic monopoles/ charges :

$$\int_V \vec{\nabla}.\vec{B} d\tau = \int_S \vec{B}.d\vec{a} = 0 \rightarrow \text{total magnetic charge contained in the volume} \tag{37}$$

Since this is true for any arbitrary volume V taken. Next we check the curl of the magnetic field :

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \times \left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) dx'\, dy'\, dz' \tag{38}$$

where one must remember that $\vec{\nabla}$ is with respect to the point $\vec{r}$. Using the vector identity:

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A}) \tag{39}$$

we have:

$$\left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}.\vec{\nabla}\right)\vec{J} - (\vec{J}.\vec{\nabla})\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} + \vec{J}\left(\vec{\nabla}.\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}\right) - \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}(\vec{\nabla}.\vec{J}) \tag{40}$$

remembering that $\vec{J}$ here is a function of $\vec{r'}$, while the $\vec{\nabla}$ acts on x, y, z, the first and last term become zero. Thus we have:

$$-(\vec{J}.\vec{\nabla})\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} + \vec{J}\left(\vec{\nabla}.\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}\right) \tag{41}$$

Now note that :

$$\vec{\nabla}.\left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}\right) = 4\pi\delta^{(3)}(\vec{r} - \vec{r'}) \tag{42}$$

While :

$$-(\vec{J}.\vec{\nabla})\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} = (\vec{J}.\vec{\nabla}')\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} \tag{43}$$

Where $\vec{\nabla}'$ is the gradient operator with respect to the coordinates x', y', z'. This is because if we have a function of seperation of two variables $f(x-x')$ then we have:

$$\frac{\partial}{\partial x}f(x-x') = -\frac{\partial}{\partial x'}f(x-x') \tag{44}$$

Plugging this in the expression for $\vec{\nabla}\times\vec{B}$ we have:

$$\begin{aligned}&\frac{\mu_0}{4\pi}\int_V\left(\vec{J}4\pi\delta^{(3)}(\vec{r}-\vec{r'}) + (\vec{J}.\vec{\nabla}')\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\right)dx'\,dy'\,dz' \\ &= \mu_0\vec{J} + \frac{\mu_0}{4\pi}\int_V(\vec{J}.\vec{\nabla}')\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}dx'\,dy'\,dz'\end{aligned} \tag{45}$$

Now the last term can be written in components. Looking at the $x-$component :

$$i\frac{\mu_0}{4\pi}\int_V(\vec{J}.\vec{\nabla}')\frac{x-x'}{|\vec{r}-\vec{r'}|^3}dx'\,dy'\,dz' \tag{46}$$

This can again be written as:

$$i\frac{\mu_0}{4\pi}\int_V\left[\vec{\nabla}'.\left(\frac{x-x'}{|\vec{r}-\vec{r'}|^3}\vec{J}\right) - \frac{x-x'}{|\vec{r}-\vec{r'}|^3}\vec{\nabla}'.\vec{J}\right]dx'\,dy'\,dz' \tag{47}$$

The second term is zero for steady flow of current since $\vec{\nabla}.\vec{J}=0$. While the first term can be written using Gauss divergence theorem:

$$i\oint_S\frac{x-x'}{|\vec{r}-\vec{r'}|^3}\vec{J}.d\vec{S}' \tag{48}$$

Now this surface can be taken as large as possible, enclosing all possible current configurations contributing to the magnetic field at the particular point. But one can always assume that there is no current source at infinity. Hence if we take boundary of the volume to infinity i.e. the suface that enclosing it to infinity, then $\vec{J}\to 0$ at infinity. Thus we can drop this term and all the other components which are identical under the replacement of $x-x'$ by $y-y'$ and $z-z'$. Thus we obtain by direct computation :

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \tag{49}$$

The Magnetic Vector potential

If we remember electrostatics, we had $\vec{\nabla}\times\vec{E}=0$ This meant we could write $\vec{E}=-\vec{\nabla}\Phi$, where Φ was our electrostatic potential. Similarly we have here $\vec{\nabla}.\vec{B}=0$. This means we can write $\vec{B}=\vec{\nabla}\times\vec{A}$, since we have :

$$\vec{\nabla}.(\vec{\nabla}\times\vec{A}) = 0 \tag{50}$$

This vector $\vec{A}$ is called the magnetic vector potential. Additionally note that the magnetic field $\vec{B}$ remains same under the transformation $\vec{A} \to \vec{A} + \vec{\nabla}\lambda$, where λ is a scalar function, as:

$$\vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}\lambda \tag{51}$$

This is called a Gauge transformation and shows the freedom in defining the vector potential $\vec{A}$. Now from the equation for curl of $\vec{B}$ we have:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \tag{52}$$

But we can use the Gauge freedom to choose $\vec{\nabla}.\vec{A} = 0$. To see this let us start with the above equation with a vector potential $\vec{A}'$ with non zero divergence:

$$\vec{\nabla}(\vec{\nabla}.\vec{A}') - \nabla^2 \vec{A}' = \mu_0 \vec{J} \tag{53}$$

Now make a gauge transformation :$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$. Plugging this in we have :

$$\vec{\nabla}(\vec{\nabla}.\vec{A}') - \nabla^2(\vec{A} + \vec{\nabla}\lambda) = \mu_0 \vec{J} \tag{54}$$

choosing λ such that :

$$\vec{\nabla}(\vec{\nabla}.\vec{A}') = \nabla^2 \vec{\nabla}\lambda = \vec{\nabla}\nabla^2\lambda \tag{55}$$

where in the last step we have interchanged $\vec{\nabla}$ and ∇^2, giving :$\vec{\nabla}(\vec{\nabla}.(\vec{A}' - \vec{\nabla}\lambda)) = \vec{\nabla}(\vec{\nabla}.\vec{A}) = 0$, which is the condition on the new vector potential $\vec{A}$. The choice on λ is then :

$$\nabla^2\lambda = \vec{\nabla}.\vec{A}' \tag{56}$$

This is Poisson's equation and hence:

$$\lambda = -\frac{1}{4\pi} \int \frac{\vec{\nabla}\vec{A}'(\vec{r}')}{|\vec{r} - \vec{r}'|} dx'\, dy'\, dz' \tag{57}$$

Where we have used $\nabla^2 |\vec{r} - \vec{r}'|^{-1} = -4\pi\delta^{(3)}(\vec{r} - \vec{r}')$. Thus starting from a $\vec{A}'$ we have constructed $\vec{A}$ whose divergence is zero by appropriately choosing λ, utilizing the gauge freedom. This means :

$$\begin{aligned} &\nabla^2 \vec{A} = -\mu_0 \vec{J} \\ &\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r})}{|\vec{r} - \vec{r}'|} dx'\, dy'\, dz' \end{aligned} \tag{58}$$

We must add a discussion here about the gauge transformation λ. Observe first:

$$\begin{aligned} \vec{\nabla}.\vec{B} \quad &= \vec{\nabla}.(\vec{\nabla} \times \vec{A}') \\ &= \vec{\nabla}.(\vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda)) \\ &= \vec{\nabla}.(\vec{\nabla} \times \vec{A}) + \vec{\nabla}.(\vec{\nabla} \times \vec{\nabla})\lambda \end{aligned} \tag{59}$$

It might seem that the last term is always zero but that might not be the case. Using the ϵ_{ijk} tensor we can write the last term as follows:

$$\epsilon_{ijk}\partial_i\partial_j\partial_k\lambda \tag{60}$$

which we can verify explicitly. Note that here i, j, k runs over $1, 2, 3$ which means x, y, z components. More over the epsilon tensor is defined as $\epsilon_{123} = 1$ and it is anti-symmetric under any interchange of indices : $\epsilon_{ijk} = -\epsilon_{ikj}$ etc. This totally defines all the components of the tensor. The question is that whether the above is always zero. To see an example we look at the easier $2-$ dimensional case :

$$\epsilon_{ij}\partial_i\partial_j\lambda \tag{61}$$

Let us integrate the above over an area :

$$\int_S dx_1 dx_2 (\partial_1\partial_2 - \partial_2\partial_1)\lambda \tag{62}$$

Please understand that you can should not use commutativity of derivatives blindly, and put the above to zero. Now let $M = \partial_2\lambda$ and $L = \partial_1\lambda$:

$$\int_S dx_1 dx_2 \left(\frac{\partial M}{\partial x_1} - \frac{\partial L}{\partial x_2}\right) \tag{63}$$

Using Green's Theorem in the plane :

$$\oint_C (L\, dx_1 + M\, dx_2) \tag{64}$$

where the integral is over the closed loop, which is the boundary of S. Thus :

$$\begin{aligned} &\oint_C \left(dx_1 \frac{\partial\lambda}{\partial x_1} + dx_2 \frac{\partial\lambda}{\partial x_2}\right) \\ &= \oint_C d\lambda \end{aligned} \tag{65}$$

Observe now that if $\lambda = \phi = \tan^{-1}\frac{x_2}{x_1}$, the polar angle and C contains the origin about which the polar angle is measured then

$$\oint_C d\phi = 2\pi \tag{66}$$

since we are going around the origin once, which is true for any arbitrary closed loop containing the origin as small as possible. This means we must have:

$$\epsilon_{ij}\partial_i\partial_j\phi = 2\pi\delta^{(2)}(\vec{r}) \tag{67}$$

Similarly one can have λ such that :

$$\epsilon_{ijk}\partial_i\partial_j\partial_k\lambda = 4\pi\delta^{(3)}(\vec{r}) \tag{68}$$

Now let us use the formula for the vector potential to compute it directly for some cases. Let us look at an infinite long current carrying wire, intersecting the y, z plane at y', z'

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I dx' \hat{i}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \tag{69}$$

note that $I dx' \hat{i} \sim \vec{J} dx' dy' dx'$ here. Integrating we have :

$$\frac{\mu_0}{4\pi} I \hat{i} \ln[(x'-x) + \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}] \Big|_{-\infty}^{\infty} \tag{70}$$

But we have to be careful in taking the limit here. To do this we replace ∞ by Λ, and then take the limit $\Lambda \to \infty$:

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \left[\frac{\mu_0}{4\pi} I \hat{i} \ln[\Lambda + \sqrt{\Lambda^2 + (y'-y)^2 + (z'-z)^2}] \right. \\
&\quad \left. - \frac{\mu_0}{4\pi} I \hat{i} \ln[-\Lambda + \sqrt{\Lambda^2 + (y'-y)^2 + (z'-z)^2}] \right] \\
&= \lim_{\Lambda\to\infty} \left[\frac{\mu_0}{4\pi} I \hat{i} \ln[\Lambda + \Lambda \sqrt{1 + \frac{(y'-y)^2 + (z'-z)^2}{\Lambda^2}}] \right. \\
&\quad \left. - \frac{\mu_0}{4\pi} I \hat{i} \ln[-\Lambda + \Lambda \sqrt{1 + \frac{(y'-y)^2 + (z'-z)^2}{\Lambda^2}}] \right] \\
&= \lim_{\Lambda\to\infty} \left[\frac{\mu_0}{4\pi} I \hat{i} \ln[\Lambda + \Lambda (1 + \frac{1}{2} \frac{(y-y')^2 + (z-z')^2}{\Lambda^2} + O(\Lambda^{-4}))] \right. \\
&\quad \left. - \frac{\mu_0}{4\pi} I \hat{i} \ln[-\Lambda + \Lambda (1 + \frac{1}{2} \frac{(y-y')^2 + (z-z')^2}{\Lambda^2} + O(\Lambda^{-4}))] \right] \\
&= \lim_{\Lambda\to\infty} \left[\frac{\mu_0}{4\pi} I \hat{i} \ln[2\Lambda + O(\Lambda^{-1}))] \right. \\
&\quad \left. - \frac{\mu_0}{4\pi} I \hat{i} \ln[\frac{1}{2} \frac{(y-y')^2 + (z-z')^2}{\Lambda} + O(\Lambda^{-3}))] \right] \\
&= -\frac{\mu_0}{4\pi} I \hat{i} \ln[(y-y')^2 + (z-z')^2] + \lim_{\Lambda\to\infty} f(\Lambda)
\end{aligned} \tag{71}$$

the term $f(\Lambda)$ diverges but is a constant and can be understood as just a choice of gauge. More over since the wire intersects the $y-z$ plane at (y', z') we have $s = \sqrt{(y-y')^2 + (z-z')^2}$, which is the radial distance from the wire. Thus :

$$\vec{A} = -\frac{\mu_0 I \hat{i}}{2\pi} \ln s \tag{72}$$

u pto a gauge transformation. One can easily check that :

$$\begin{aligned}
\vec{\nabla}\times\vec{A} &= \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix} \\
&= \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\mu_0 I}{2\pi}\ln s \end{vmatrix} \\
&= \frac{\mu_0 I}{2\pi s}\hat{\phi}
\end{aligned} \tag{73}$$

where we have taken the wire along the z axis to match with the standard form of the curl in cylindrical coordinates.

Magnetix Flux

Now consider the following integral :

$$\oint_C \vec{A}.d\vec{l} \tag{74}$$

which is the line integral of the vector potential along a closed loop C. We can use Stoke's theorem on this :

$$\oint_C \vec{A}.d\vec{l} = \int_S (\vec{\nabla}\times\vec{A}).d\vec{S} = \int_S \vec{B}.d\vec{S} \tag{75}$$

Which is the surface integral of $\vec{B}$ over any surface S bounded by C. Thus although S is not unique, the integral is always the same. This is called the Magnetic flux Φ. This has similar application to the Ampere's law as one can use this to find the vector potential in many configurations. As an example let us find the vector potential due to a solenoid of radius a : If we take a circular loop out side the solenoid , perpendicular to the axis of the solenoid , then $\vec{B}=\vec{0}$ outside, $\vec{B}=\mu_0 nI\hat{z}$ inside. Using the line integral over $\vec{A}$:

$$\oint_C \vec{A}.d\vec{l} = \int_0^{2\pi} A_\phi\hat{\phi}.\hat{\phi}s d\phi = A_\phi s 2\pi \tag{76}$$

where we have used the fact that the loop traverses in the ϕ direction hence only that component of $\vec{A}$ contributes, and by symmetry around the cylinder A_ϕ can not be a function of ϕ. While from the area integral of $\vec{B}$ we have:

$$\begin{aligned}
&\int_S \vec{B}.d\vec{S} = \int_0^a\int_0^{2\pi} \vec{B}_{inside}.\hat{z}\rho\, d\rho\, d\phi + \int_a^s\int_0^{2\pi} \vec{B}_{outside}.\hat{z}\rho\, d\rho\, d\phi \\
&= \int_0^a\int_0^{2\pi} \mu_0 nI\hat{z}.\hat{z}\rho\, d\rho\, d\phi + \int_a^s\int_0^{2\pi} \vec{0}.\hat{z}\rho\, d\rho\, d\phi \\
&= \mu_0 nI\pi a^2
\end{aligned} \tag{77}$$

Finally equating we have :

$$\begin{aligned} A_\phi s 2\pi &= \mu_0 n I \pi a^2 \\ \Rightarrow A_\phi &= \frac{\mu_0 n I a^2}{2s} \end{aligned} \tag{78}$$

This is the vector potential outside the solenoid. Note that although the magnetic field is zero outside, this does not mean that $\vec{A}$ is zero outside. Next if we take the loop inside, the line integral over $\vec{A}$ is identical, while the surface integral over $\vec{B}$ yields :

$$\int_S \vec{B}.d\vec{S} = \int_0^s \int_0^{2\pi} \mu_0 n I \hat{z}.\hat{z}\rho d\rho d\phi = \mu_0 n I s^2 \tag{79}$$

equating, wefind the vector potential inside :

$$A_\phi = \frac{\mu_0 n I s}{2} \tag{80}$$

Problems and discussion

8. The magnetic vector potential of a uniform magnetic field is given by $\vec{A} = \frac{\vec{B}\times\vec{r}}{2}$. One can explicitly check this :

$$\begin{aligned} \vec{\nabla}\times\vec{A} &= \frac{1}{2}\vec{\nabla}\times(\vec{\nabla}\times\vec{r}) \\ &= \frac{1}{2}[\vec{B}(\vec{\nabla}.\vec{r}) - \vec{r}(\vec{\nabla}.\vec{B}) + (\vec{r}.\vec{\nabla})\vec{B} - (\vec{B}.\vec{\nabla})\vec{r}] \end{aligned} \tag{81}$$

Noting that

$$\begin{aligned} &\vec{\nabla}.\vec{B} = 0 \\ &\vec{\nabla}.\vec{r} = 3 \\ &(\vec{r}.\vec{\nabla})\vec{B} = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = 0 \\ &(\vec{B}.\vec{\nabla})\vec{r} = (B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z})(\hat{i}x + \hat{j}y + \hat{k}z) = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z = \vec{B} \end{aligned} \tag{82}$$

Plugging this in we have:

$$\frac{1}{2}[3\vec{B} - \vec{B}] = \vec{B} \tag{83}$$

we can check the gauge condition :

$$\begin{aligned} \vec{\nabla}.\vec{A} &= \frac{1}{2}\vec{\nabla}.(\vec{B}\times\vec{r}) \\ &= \frac{1}{2}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} \\ &= 0 \end{aligned} \tag{84}$$

We can use this to find the Magnetic field due to surface current $\vec{K} = K\hat{i}$, which is given by:

$$
\begin{aligned}
\vec{B} &= -\frac{\mu_0 K}{2}\hat{j} \quad z > 0 \\
&= \frac{\mu_0 K}{2}\hat{j} \quad z < 0
\end{aligned} \tag{85}
$$

plugging it in the form of the vector potential :

$$
\begin{aligned}
\vec{A} &= \frac{\vec{B} \times \vec{r}}{2} \\
&= -\frac{\mu_0 K}{4}(\hat{j} \times \vec{r}) \\
&= -\frac{\mu_0 K}{4}(-x\hat{k} + z\hat{i}) \quad z > 0
\end{aligned} \tag{86}
$$

analogously we can find the case for $z < 0$.

9. Multi-pole Expansion : Given a current configuration flowing along some closed loop C, we have the vector potential :

$$
\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint_C \frac{I\, d\vec{l}}{|\vec{r} - \vec{r}'|} \tag{87}
$$

Now $|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'}$, θ' being the angle between $\vec{r}$ and $\vec{r}'$. Now if the point of observation $\vec{r}$ is far away from the current configuration, then $|\vec{r}| = r > |\vec{r}'| = r'$. Thus we can use the expansion corresponding to Legendre polynomial

$$
\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} = \frac{1}{r\sqrt{1 - 2\frac{r'}{r}\cos\theta' + (\frac{r'}{r})^2}} = \frac{1}{r}\sum_{n=0}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \tag{88}
$$

Plugging this in the expression for the vector potential :

$$
\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint_C \frac{1}{r}\sum_{n=0}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')d\vec{l} \tag{89}
$$

Interchanging order of integration and summation, under assumption of convergence we have:

$$
\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}\frac{1}{r^{n+1}}\oint_C (r')^n P_n(\cos\theta')d\vec{l} \tag{90}
$$

This is very similar to the expansion in electrostatics. The first term is the monopole term corresponding to r^{-1} i.e. $n = 0$. That is zero since we have a closed loop integral :

$$
\oint_C d\vec{l} = 0 \tag{91}
$$

The next term is the dipole term corresponding to r^{-2}, and so on. Let us look at this term :

$$\begin{aligned} &\frac{\mu_0 I}{4\pi r^2}\oint_C r'\cos\theta' d\vec{l} \\ &= \frac{\mu_0 I}{4\pi r^2}\oint_C (\hat{r}.\vec{r}')d\vec{l} \\ &= \frac{\mu_0 I}{4\pi r^2}\int d\vec{a}\times\hat{r} \end{aligned} \tag{92}$$

where we have used the vector identity :

$$\int \vec{\nabla} T \times d\vec{a} = -\oint_C T d\vec{l} \tag{93}$$

Plugging in $T = \hat{r}.\vec{r}'$, where $\hat{r}$ is a given constant vector and $\vec{\nabla}$ is with respect to $\vec{r}'$, we find:

$$\int d\vec{a}\times\hat{r} = \oint_C (\hat{r}.\vec{r}')d\vec{l} \tag{94}$$

Now $\int d\vec{a}$ is the total area of the loop. Defining $\vec{m} = I\int d\vec{a}$ as the magnetic moment , we finally obtain the magnetic vector potential for a dipole:

$$\vec{A} = \frac{\mu_0}{4\pi r^2}(\vec{m}\times\hat{r}) \tag{95}$$

Using this we can calculate the magnetic field due to the dipole simply by taking the curl of the above. Let us assume the $\vec{m}$ is along the z-axis, then $\vec{m}\times\hat{r} = m\sin\theta\hat{\phi}$, which gives us :

$$\begin{aligned} \vec{\nabla}\times\vec{A} \quad &= \vec{\nabla}\times\frac{\mu_0}{4\pi r^2}m\sin\theta\hat{\phi} \\ &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & (r\sin\theta)\frac{\mu_0}{4\pi r^2}m\sin\theta \end{vmatrix} \\ &= \hat{r}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0}{4\pi r^2}mr\sin^2\theta\right) - \frac{1}{r^2\sin\theta}r\hat{\theta}\frac{\partial}{\partial r}\left(\frac{\mu_0}{4\pi r^2}mr\sin^2\theta\right) \\ &= \hat{r}\frac{\mu_0}{4\pi r^3}2\cos\theta + \hat{\theta}\frac{\mu_0}{4\pi r^3}\sin\theta \end{aligned} \tag{96}$$

Noting that $\vec{m}.\hat{r} = m\cos\theta$ and $\vec{m}.\hat{\theta} = -m\sin\theta$, we can rewrite: $2m\cos\theta\hat{r} + m\sin\theta\hat{\theta} = 3(\vec{m}.\hat{r}) - \vec{m}$. Hence we have the magnetic field for a dipole :

$$\vec{B} = \frac{\mu_0}{4\pi r^3}[3(\vec{m}.\hat{r}) - \vec{m}] \tag{97}$$

10. Find the magnetic field due to a constant vector potential $\vec{A} = A_\phi\hat{\phi}$

11. Find the magnetic field due to the vector potential $\vec{A} = \phi\hat{z}$, where ϕ is the azimuthal angle around the axis of the cylinder in $z-$direction. Also show that $\vec{\nabla}.\vec{B}$ is non zero in this case.

12. Find the magnetic field due to a circular disc of radius R spinning with angular velocity ω about its $z-$ axis at height h from its center on the axis.
13. Solve $\vec{\nabla} \times \vec{A} = \vec{B}$ directly. Let us do this problem : Remember that we had the Biot-Savart law, with the steady state current condition $\vec{\nabla}.\vec{J} = 0$, which gives us the form of the magnetic field in terms of $\vec{J}$:

$$\vec{B}(\vec{r}) = \frac{1}{4\pi}\int (\mu_0 \vec{J}(\vec{r}')) \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dx'\, dy'\, dz' \tag{98}$$

which is the solution to the differential equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$. Now we have the differential equation $\vec{\nabla} \times \vec{A} = \vec{B}$ with the identical condition $\vec{\nabla}.\vec{B} = 0$. Thus the solution can be carried over simply by substituting in Biot-Savart's : $\vec{B} \to \vec{A}$ and $\mu_0 \vec{J} \to \vec{B}$:

$$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int \vec{B}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dx'\, dy'\, dz' \tag{99}$$

14. Given a semi-circular loop in the lower half plane with current I flowing in the counter clockwise dirction find the magnetic field at any point on the corresponding semi-circle in the upper half plane. Given the radius of the circle is R. Now this can be computed directly using Biot-Savart's law, where $\vec{r}' = R\cos\phi\hat{i} - R\sin\phi\hat{j}$ the point parameterizing the semi-circular loop over which the current flows in the lower half plane, and $\vec{r} = R\cos\theta\hat{i} + R\sin\theta\hat{j}$ parameterizing the points at which we need to find $\vec{B}$. Here θ and ϕ are measured with respect to the positive $x-$axis. Biot-savart's law reads :

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int d\vec{l} \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} dx'\, dy'\, dz' \tag{100}$$

Here $d\vec{l}$ in is the variation along the semi-circular loop parametrized by $\vec{r}'$. Since R is constant, variation is only along ϕ, thus :

$$d\vec{l} = d\vec{r}'|_{R=constant} = -R\sin\phi\, d\phi\hat{i} - R\cos\phi\, d\phi\hat{j} = -[R\sin\phi\hat{i} + R\cos\phi\hat{j}]d\phi \tag{101}$$

Hence we have :

$$\begin{aligned} d\vec{r'} \times (\vec{r}-\vec{r'}) &= -\begin{vmatrix} i & j & k \\ R\sin\phi\, d\phi & R\cos\phi\, d\phi & 0 \\ R(\cos\theta - \cos\phi) & R(\sin\theta + \sin\phi) & 0 \end{vmatrix} \\ &= -\hat{k}d\phi R^2[\sin\phi\sin\theta + \sin^2\phi - \cos\theta\cos\phi + \cos^2\phi] \\ &= -\hat{k}R^2 d\phi[1 - \cos(\theta+\phi)] \end{aligned} \tag{102}$$

Plugging this in for $\vec{B}$ we have:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_\pi^0 \frac{-[1-\cos(\theta+\phi)]\hat{k}R^2 d\phi}{[R^2(\cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi) + R^2(\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi)]^{3/2}} \tag{103}$$

Note that the integral is from π to 0 signifying the direction of flow of current along the element $d\vec{r'}$. Simplifying :

$$
\begin{aligned}
&= \frac{\mu_0 I}{4\pi} \int_0^{\pi} \frac{[1 - \cos(\theta + \phi)]\hat{k} R^2 d\phi}{[2R^2(1 - \cos(\theta + \phi)]^{3/2}} \\
&= \frac{\mu_0 I}{16\pi R} \hat{k} \int_0^{\pi} \frac{d\phi}{\sin \frac{\theta + \phi}{2}} \\
&= \frac{\mu_0 I}{8\pi R} \hat{k} \ln \left[\frac{\tan \frac{\theta + \pi}{4}}{\tan \frac{\theta}{4}} \right]
\end{aligned}
\tag{104}
$$

15. Given a dipole $\vec{m} = -m_0 \hat{k}$ and a constant magnetic field $\vec{B} = B_0 \hat{k}$, find the radius of the sphere with no radial component of magnetic field. For this let us write the total magnetic field due to the back ground and the dipole :

$$\vec{B} = B_0 \hat{k} - \frac{\mu_0 m_0}{4\pi r^3} [2 \cos\theta \vec{r} + \sin\theta \hat{\theta}] \tag{105}$$

Since we want $\vec{B}$ to be perpendicular $\hat{r}$ i.e. component in radial direction is zero, we have:

$$\vec{B}.\hat{r} = B_0 \hat{k}.\hat{r} - \frac{\mu_0 m_0}{4\pi r^3} [2 \cos\theta \vec{r} + \sin\theta \hat{\theta}].\hat{r} = 0 \tag{106}$$

Note that $\hat{k}.\hat{r} = \cos\theta$, hence :

$$[B_0 - \frac{\mu_0 m_0}{4\pi r^3} 2] \cos\theta = 0 \tag{107}$$

Since this is valid for any value of the θ, we must have:

$$r^3 = \frac{\mu_0 m_0}{2\pi B_0} \tag{108}$$

This gives the sphere of no radial component of magnetic field.