Subject Name - PHYSICS

Semester - (IV)

Name of the Teacher - Bidhan Chandra Jana

Name of the Topic - Alternating Currents.

Alternating Currents

AC response of a Resistance, an Inductance, and a Capacitance -

Resistance -

Suppose a voltage $v(t) = V_0 \sin\omega t$ is applied across a resistance R. The instantaneous current -

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R}\sin\omega t = I_0 \sin\omega t$$

where, $I_0 = \frac{V_0}{R}$.

Here $i(t)$ is a sinusoidal alternating current with the same frequency as $v(t)$ and in phase with it.

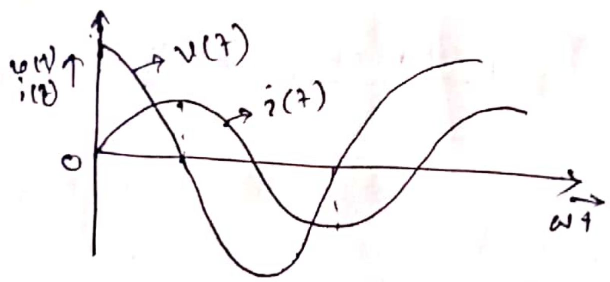
Inductance -

Let a sinusoidal current $i(t) = I_0 \sin\omega t$ pass through an inductance L. The instantaneous back emf induced across the inductance will be $-L\frac{di}{dt}$.

$\therefore$ the voltage across L is given by

$$v(t) = L\frac{di}{dt} = L\frac{d}{dt}(I_0 \sin\omega t)$$
$$= \omega L I_0 \cos\omega t$$
$$= \omega L I_0 \sin\left(\omega t + \frac{\pi}{2}\right).$$

$\therefore$ The voltage across an inductance leads the current $i(t)$ through $\frac{\pi}{2}$ or $90°$.



Variation of $v(t)$ and $i(t)$ for the inductance

Capacitance -

A voltage $v(t) = V_0 \sin\omega t$ is applied across a capacitor.

The instantaneous charge stored in the capacitor is $q(t) = Cv(t)$

C is the capacitance of the capacitor.

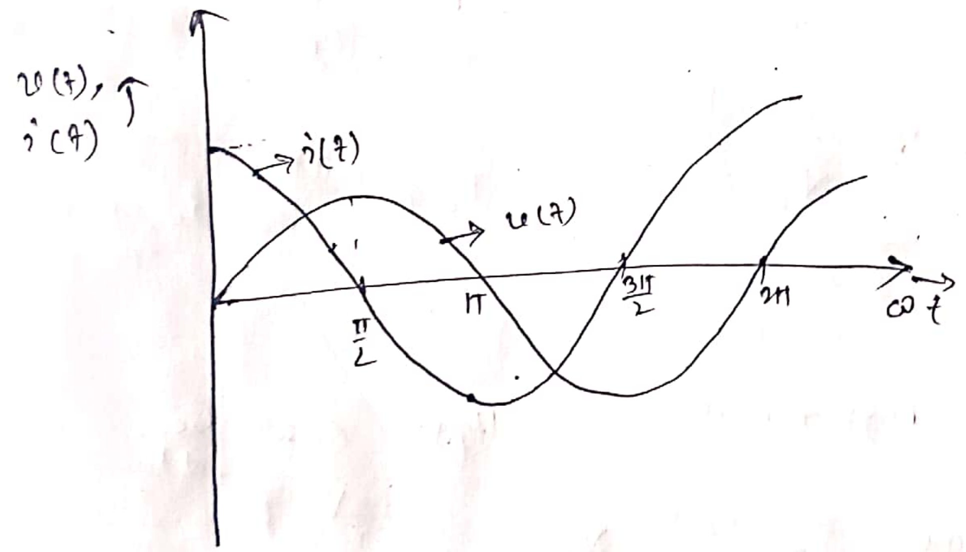
∴ The instantaneous current in the circuit is :-

$$i(t) = \frac{dq}{dt} = C\frac{dv}{dt}$$

$$= \omega C V_0 \cos\omega t$$

$$\Rightarrow \quad i(t) = \omega C V_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

∴ The current $i(t)$ leads the voltage $v(t)$ across the capacitor by $\pi/2$ or $90°$.



Variation of $v(t)$ and $i(t)$ with ωt of the capacitor.

Root-Mean-Square (RMS) value of Current :-

$$i(t) = I_0 \sin \omega t$$

The rms value of a sinusoidal current is, given by

$$I_{rms}^2 = \frac{\int_0^T i^2(t)\, dt}{\int_0^T dt}$$

$$= \frac{I_0^2}{T} \int_0^{2\pi} \sin^2 \omega t \, dt$$

$$= \frac{I_0^2}{T} \int_0^T \frac{1 - \cos 2\omega t}{2} \, dt.$$

$$= \frac{I_0^2}{2T} \left[t - \frac{\sin 2\omega t}{2\omega} \right]_0^T.$$

$$= \frac{I_0^2}{2}$$

$$\Rightarrow \boxed{I_{rms} = \frac{I_0}{\sqrt{2}}}$$

Similarly, rms value of voltage can be written as $\boxed{V_{rms} = \frac{V_0}{\sqrt{2}}}$.

Average value of current :-

The average value of a sinusoidal current or voltage over a full cycle is zero. However, it is not zero over a half cycle.

Average for a current $i(t) = I_0 \sin\omega t$ for a half cycle is given by —

$$\omega = \frac{2\pi}{T}$$

$$\omega T = 2\pi$$

$$I_{av} = \frac{\int_0^{T/2} i(t)\,dt}{\int_0^{T/2} dt}$$

$$= \frac{2I_0}{T} \int_0^{T/2} \sin\omega t\,dt$$

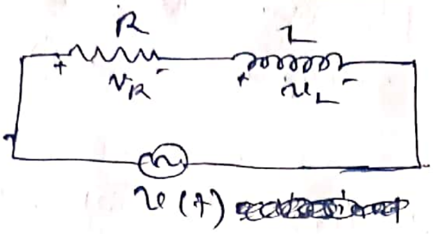
$$= \frac{2I_0}{T} \left[-\frac{\cos\omega t}{\omega}\right]_0^{T/2}$$

$$= \frac{2I_0}{T\omega} \left[-\cos\frac{\omega T}{2} + \cos 0\right]$$

$$= \frac{2I_0}{2\pi} [1+1]$$

$$\Rightarrow \boxed{I_{av} = \frac{2I_0}{\pi}}$$

AC voltage applied to a Series LR circuit :-



The emf equation for the circuit is –

$$v_R(t) + v_L(t) = v(t)$$

$$\Rightarrow \quad Ri + L\frac{di}{dt} = v(t)$$

$$\Rightarrow \quad Ri + L\frac{di}{dt} = V_0 e^{j\omega t} \quad \text{--- (1)}$$

[assuming the instantaneous voltage $v(t) = V_0 e^{j\omega t}$ $= Re[V_0 e^{j\omega t}]$]

At steady state current will follow the voltage.

∴ Steady state solution for current can be written as – $i(t) = I_0 e^{j\omega t}$, I_0 is complex quantity.

Now, substituting in eqn (1) –

$$RI_0 e^{j\omega t} + L\frac{d}{dt}(I_0 e^{j\omega t}) = V_0 e^{j\omega t}$$

$$\Rightarrow \quad RI_0 + j\omega L I_0 = V_0$$

$$\Rightarrow \quad I_0 = \frac{V_0}{R + j\omega L} = \frac{V_0}{Z}$$

$Z = R + j\omega L$ is known as impedance of the circuit.

$$|Z| = \sqrt{R^2 + \omega^2 L^2}$$

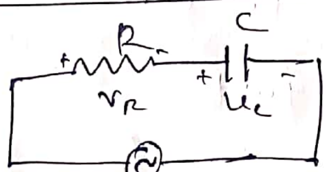
and $\tan\theta = \frac{\omega L}{R}$; $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$

$\therefore$ Impedance, $Z = |Z| e^{j\theta}$

$\therefore$ The current $i(t)$ is given by

$$\boxed{i(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\omega t - \theta)}}$$

AC applied to a series RC circuit :-



$+ \quad -$

$V(t) = V_0 e^{j\omega t}$

The emf equation for the circuit is —

$$v_R(t) + V_C(t) = V(t).$$

$$\Rightarrow \quad Ri + \frac{q}{C} = V_0 e^{j\omega t} \quad \text{---} (1)$$

$i = \frac{dq}{dt}$

$q = \int i \, dt$

At steady state, the current in the circuit can be written as $i(t) = I_0 e^{j\omega t}$

I_0 is complex.

$\therefore$ From eqn (1), substituting $i(t)$.

$$R I_0 e^{j\omega t} + \frac{1}{C} \int I_0 e^{j\omega t} dt = V_0 e^{j\omega t}$$

$$\Rightarrow R I_0 e^{j\omega t} + \frac{I_0}{C} \frac{e^{j\omega t}}{j\omega} = V_0 e^{j\omega t}$$

$$\Rightarrow I_0 = \frac{V_0}{R + \frac{1}{j\omega C}} = \frac{V_0}{R - \frac{j}{\omega C}} = \frac{V_0}{Z}$$

Where, $Z = R - \frac{j}{\omega C}$ is known as impedance of the circuit.

$$|Z| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$$

and $\tan\theta = -\frac{1}{\omega C R}$

$$\therefore Z = R - \frac{j}{\omega C} = |Z| e^{-j\theta}$$

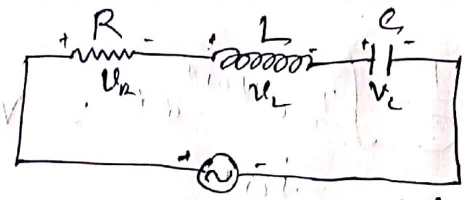
$\therefore$ The current in the circuit –

$$i(t) = \frac{V_0}{Z} e^{j\omega t}$$

$$= \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} e^{j(\omega t + \theta)}$$

$$\Rightarrow \boxed{i(t) = I_0 e^{j(\omega t + \theta)}}$$

AC to a Series LCR Circuit :-



$V(t) = V_0 e^{j\omega t}$

The emf equation for the circuit is

$$V_R(t) + V_L(t) + V_C(t) = V(t)$$

$$\Rightarrow Ri + L\frac{di}{dt} + \frac{q}{C} = V_0 e^{j\omega t} \quad \text{——} \; (1)$$

⇒ At steady state, the solution for the current is given by $i(t) = I_0 e^{j\omega t}$, I_0 is complex.

∴ From eqn (1), substituting $i(t)$

$$R I_0 e^{j\omega t} + L I_0 \frac{d}{dt}(e^{j\omega t}) + \frac{1}{C}\frac{I_0 e^{j\omega t}}{j\omega} = V_0 e^{j\omega t} \qquad \left[\because q = \int i\,dt = \frac{I_0 e^{j\omega t}}{j\omega}\right]$$

$$\Rightarrow R I_0 + j\omega L I_0 + \frac{I_0}{j\omega C} = V_0$$

$$\Rightarrow \boxed{I_0 = \frac{V_0}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{V_0}{Z}}$$

Here, $Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$ is called impedance of the circuit.

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and $\tan\theta = \dfrac{\omega L - \frac{1}{\omega C}}{R}$

$$\therefore \quad Z = |Z| e^{j\theta}$$

∴ The current in the circuit is —

$$\boxed{i(t) = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\, e^{j(\omega t - \theta)}}$$

Series Resonance :-

The impedance of the series LCR circuit at an angular frequency ω of the applied voltage is

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

at resonance in the circuit, the reactance term must be zero and current (I_0) becomes maximum.

$$\therefore \omega L - \frac{1}{\omega C} = 0$$

$$\Rightarrow \omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\therefore f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

f_0 is the resonant frequency.

At resonance, the circuit is purely resistive. The impedance at resonance is R, and the maximum value of current becomes $I_0 = \frac{V_0}{R}$.

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