Subject Name - Mathematics
Semester - IV
Name of the teacher - Milan Kumar Jana
Name of the topic - Determinants

Q 1. With out expanding prove that

$$\begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$$

Ans Let $\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix}$

$$\Rightarrow -\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix}$$

$$\Rightarrow -\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix}$$ interchanging Row and Columns

$\Rightarrow -\Delta = \Delta$

$\Rightarrow 2\Delta = 0$

$\Rightarrow \Delta = 0$

2. Without expanding the determinant, prove that

$$\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix} = 0.$$

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}.$$

Then $\Delta = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \Delta_1 + \Delta_2$, say.

Now $\Delta_2 = \frac{1}{abcd} \begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$ [multiplying R_1 by a, R_2 by b, R_3 by c, R_4 by d]

$= \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$ [multiplyng c_4 by $\frac{1}{abcd}$]

$= - \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$ [applying three successive interchanges to bring C_4 to C_1]

$= -\Delta_1$. Therefore $\Delta = 0$.

Worked Examples.

1. Prove that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

We have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[interchanging R_2 and R_3 in the right hand determinant]

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}.$$

[multiplication of rows by columns]

Therefore

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

উদাহরণ 16. বিস্তৃত না করে প্রমাণ করো যে, $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix} = 0$।

সমাধান : বামপক্ষ $= \begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & ab & abc\cdot\frac{b+a}{ab} \\ 1 & bc & abc\cdot\frac{c+b}{bc} \\ 1 & ca & abc\cdot\frac{a+c}{ca} \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & ab & bc+ca \\ 1 & bc & ca+ab \\ 1 & ca & ab+bc \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} 1 & ab & ab+bc+ca \\ 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \end{vmatrix} \quad [C_3' = C_3 + C_2]$$

$$= \frac{ab+bc+ca}{abc}\begin{vmatrix} 1 & ab & 1 \\ 1 & bc & 1 \\ 1 & ca & 1 \end{vmatrix} = \frac{ab+bc+ca}{abc} \times 0$$

[∵ প্রথম ও তৃতীয় স্তম্ভ অভিন্ন]

$= 0$ *(প্রমাণিত)*।

উদাহরণ 13. $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$ নির্ণায়কের মান নির্ণয় করো এবং তারপর দেখাও যে,

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2 \text{।}$$

সমাধান : মনে করা হল, $D = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

D নির্ণায়ককে প্রথম সারি সাপেক্ষে বিস্তৃত করে,

$$D = -c\begin{vmatrix} c & a \\ b & 0 \end{vmatrix} + b\begin{vmatrix} c & 0 \\ b & a \end{vmatrix} = -c(0-ab)+b(ca-0) = 2abc$$

এখন, D নির্ণায়কের অ্যাডজুগেট্ D' হলে,

$$D' = \begin{vmatrix} \begin{vmatrix} 0 & a \\ a & 0 \end{vmatrix} & -\begin{vmatrix} c & a \\ b & 0 \end{vmatrix} & \begin{vmatrix} c & 0 \\ b & a \end{vmatrix} \\ -\begin{vmatrix} c & b \\ a & 0 \end{vmatrix} & \begin{vmatrix} 0 & b \\ b & 0 \end{vmatrix} & -\begin{vmatrix} 0 & c \\ b & a \end{vmatrix} \\ \begin{vmatrix} c & b \\ 0 & a \end{vmatrix} & -\begin{vmatrix} 0 & b \\ c & a \end{vmatrix} & \begin{vmatrix} 0 & c \\ c & 0 \end{vmatrix} \end{vmatrix} \quad \text{বা, } D' = \begin{vmatrix} -a^2 & ab & ca \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix}$$

বা, $$D' = \begin{vmatrix} -a^2 & ab & ca \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix} = D^2 \quad [\because D' = D^2]$$

$$= (2abc)^2 = 4a^2b^2c^2$$ (প্রমাণিত)।

উদাহরণ 10. x-এর মান নির্ণয় করো যখন $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x+1 & -1 \\ -1 & 1 & x+1 \end{vmatrix} = 0$ ।

সমাধান : $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x+1 & -1 \\ -1 & 1 & x+1 \end{vmatrix} = 0$ বা, $\begin{vmatrix} x & 0 & 1 \\ 0 & x+2 & -1 \\ x & -x & x+1 \end{vmatrix} = 0$ $[C_1' = C_1 + C_3$, $C_2' = C_2 - C_3]$

$$\begin{vmatrix} x & 0 & 1 \\ 0 & x+2 & -1 \\ 0 & -x & x \end{vmatrix} = 0 \qquad [R_3' = R_3 - R_1]$$

$x\begin{vmatrix} x+2 & -1 \\ -x & x \end{vmatrix} = 0$ বা, $x(x^2 + 2x - x) = 0$ বা, $x^2(x+1) = 0$

$x = 0$ বা, $x = -1$

...এর বর্গকে একটি তৃতীয় ক্রমের