Energy of a vibrating String:-

Now We know that when a stretched string is rigidly fixed at both ends, then displacement

$$y = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos(\omega_n t - \phi_n). \longrightarrow (1)$$

Let $Y_n = C_n \cos(\omega_n t - \phi_n) \rightarrow (2).$

then $\dot{Y}_n = -\omega_n C_n \sin(\omega_n t - \phi_n) \longrightarrow (3)$

$$\therefore \dot{y} = -\sum \sin \frac{n\pi x}{l} \dot{Y}_n \longrightarrow (4)$$

So, the total K.E. of the string,

$$K = \frac{1}{2}\int_0^l m\dot{y}^2\,dx = \frac{m}{2}\int_0^l \left(-\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\dot{Y}_n\right)^2 dx$$

$$= \frac{m}{2}\int_0^l \sum_{n=1}^{\infty}\sum_{m=1} \dot{Y}_n \dot{Y}_m \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$

$$= \frac{m}{2}\int_0^l \left(\sum \dot{Y}_n^2 \sin^2\frac{n\pi x}{l} + 2\sum_{n\neq m} \dot{Y}_n\dot{Y}_m \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\right) \rightarrow (5).$$

Now, $\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 0$ when $n\neq m$

$= \frac{l}{2}$ when $n = m$.

$$\therefore K = \frac{m}{2}\cdot\frac{l}{2}\sum \dot{Y}_n^2 = \frac{M}{4}\sum \dot{Y}_n^2$$

$$= \frac{M}{4}\sum \omega_n^2 c_n^2 \sin^2(\omega_n t - \phi_n) \rightarrow (6).$$

where M is the total mass of the string $M = m$

The P.E associated with an element dx of the string is equal to the work done in stretching this element to the length $ds = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,dx = \sqrt{(\delta x)^2+(\delta}$

As the tension T is supposed to remain const this work is, $dP = (T\,ds - T\,dx)$

$$dP = T\left[\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}^{1/2} dx - dx\right] = \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx. \rightarrow (7)$$

$\because \frac{\partial y}{\partial x} << 1 \quad \therefore$ from (1). $\frac{\partial y}{\partial x} = \sum \frac{n\pi}{l}\cos\frac{n\pi x}{l}$.

$\therefore$ Total P.E.

$$P = \int dP = \int_0^l \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{T}{2}\int_0^l \sum \frac{n^2\pi^2}{l^4} Y_n^2 \cos^2\frac{n\pi x}{l}\,dx \rightarrow (8)$$

$$= \frac{T}{2} \sum \frac{n^2\pi^2}{l^2} \frac{l}{2} Y_n^2 \quad \left[\because \int_0^l \cos^2\frac{n\pi x}{l} dx = \frac{l}{2}\right]$$

$$= \frac{M}{4} \sum \omega_n^2 Y_n^2 = \frac{M}{4} \sum \omega_n^2 C_n^2 \cos^2(\omega_n t - \phi_n) \rightarrow (9)$$

$\therefore$ Total energy

$$E = K + P = \frac{M}{4}\left(\sum \dot{Y}_n^2 + \sum \omega_n^2 Y_n^2\right) = \frac{M}{4} \sum \omega_n^2 C_n^2 \rightarrow (10)$$

Evaluation of consts a_n and b_n

$$y = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos(\omega_n t - \phi_n)$$

OR

$$y = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} (a_n \cos\omega_n t + b_n \sin\omega_n t) \rightarrow (1).$$

$$\dot{y}(x,t) = \sum \omega_n \sin\frac{n\pi x}{l} (-a_n \sin\omega_n t + b_n \cos\omega_n t). \rightarrow (2)$$

The values of y and $\dot{y}$ at $t=0$. obtained by putting $t=0$ in (1) and (2). This gives

$$y_0(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}. \rightarrow (3)$$

and

$$\dot{y}_0(x,0) = \sum \omega_n b_n \sin\frac{n\pi x}{l}.$$

$$= \left(\frac{nc}{l}\right) \sum n b_n \sin\frac{n\pi x}{l}. \rightarrow (4).$$

These are the expansions of the functions $y(x,0)$ in terms of the eigenfunctions $\sin\frac{n\pi x}{l}$.

Multiplying both sides of (3) by $\sin\frac{m\pi x}{l}$ where m has any integral value 1, 2, 3 etc. and integrating from $x=0$ to $x=l$,

$$\int_0^l y_0 \sin\frac{m\pi x}{l}\,dx = \int_0^l \sum a_n \sin\frac{n\pi x}{l}\,\sin\frac{m\pi x}{l}\,dx$$

The R.H.S. vanishes for all values of $n \neq m$ only some value exists when $m = n$.

$$\therefore \int_0^l y_0 \sin\frac{n\pi x}{l}\,dx = a_n\int_0^l \sin^2\frac{n\pi x}{l}\,dx = a_n\cdot\frac{l}{2}$$

$$\text{or, } a_n = \frac{2}{l}\int_0^l y_0 \sin\frac{n\pi x}{l}\,dx \longrightarrow (5)$$

Multiplying (4) by $\sin\frac{m\pi x}{l}\,dx$ and proceeding in a similar way,

$$b_n = \frac{2}{n\pi c}\int_0^l \dot{y}_0 \sin\frac{n\pi x}{l}\,dx.$$

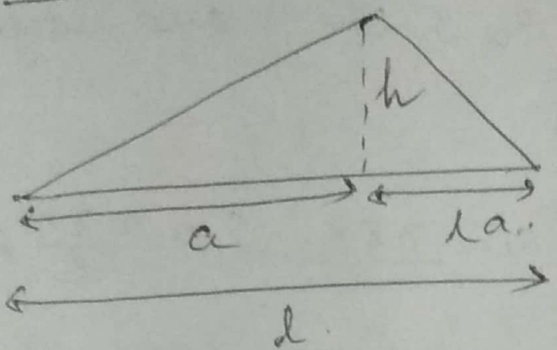
The values of a_n and b_n will naturally depend on the mode of excitation of the string.

These are (1) The plucked string. (sitar or banjo)

(2). The struck string (piano)

(3). The bowed string. (esraj or violin)

Plucked String :-



Let, a stretched string of length l be plucked at a point distant a from one end (which is origin), so that the displacement at a is h. h is so small, that the tension is supposed to remain const.

The initial configuration of the string (from geometry).

$$\frac{y_0}{x} = \frac{h}{a}.$$

$$y_0 = \frac{hx}{a} \quad \text{from } x=0 \text{ to } x=a.$$

and

$$\frac{y_0}{(l-x)} = \frac{h}{(l-a)} \quad \text{from } x=a \text{ to } x=l. \rightarrow (1)$$

The initial velocity everywhere is zero.

$$\dot{y}_0 = 0.$$

Now $y_0 = \sum a_n \sin \frac{n\pi x}{l}.$

$$\dot{y}_0 = \sum \omega_n b_n \sin \frac{n\pi x}{l}.$$

since $\dot{y}_0 = 0$; the co-eff b_n are all zero.

So $y(x,t)$ reduces to,

$$y(x,t) = \sum a_n \sin \frac{n\pi x}{l} . \cos \omega_n t.$$

again $a_n = \frac{2}{l} \int_0^l y_0 \sin \frac{n\pi x}{l} dx.$

$$= \frac{2}{l}\left[\int_0^a \left(\frac{hx}{a}\right) \sin \frac{n\pi x}{l} dx + \int_a^l \frac{h(l-x)}{(l-a)} \sin \frac{n\pi x}{l} dx\right]$$

take, $k = \frac{n\pi}{l}$,

$\therefore \int x \sin kx \, dx = -x \frac{\cos kx}{k} + \int \frac{\cos kx}{k} dx.$

(first term)

$= -\frac{x}{k} \cos kx + \frac{\sin kx}{k^2}$

and

$\int (l-x) \sin kx \, dx = (l-x) \frac{-\cos kx}{k} - \int \frac{\cos kx}{k} dx.$

$= -\frac{(l-x)}{k} \cos kx - \frac{\sin kx}{k^2}$ (second term)

$\therefore a_n = \frac{2}{l}\left[\left[\left(\frac{h}{a}\right)\left(-\frac{x}{k}\right) \cos kx + \frac{h}{a}\frac{\sin kx}{k^2}\right]_0^a + \left[-\frac{h}{(l-a)}\frac{(l-x)}{k} \cos kx + \frac{h}{(l-a)}\frac{\sin kx}{k^2}\right]_a^l\right]$

$= \frac{2}{l}\left[\left[\cancel{\frac{h}{a}\left(-\frac{a}{k}\right) \cos \frac{n\pi}{l} a} + \frac{h}{a}\frac{\sin ka}{k^2}\right] + \frac{2}{l}\left[\frac{h}{(l-a)}\frac{(l-l)}{k} \cos \frac{n\pi}{l} + \frac{h}{(l-a)}\frac{\sin kl}{k^2} + \cancel{\frac{h}{(l-a)}\frac{(l-a)}{k} \cos \frac{n\pi}{l} a} - \frac{h}{(l-a)}\frac{\sin ka}{k^2}\right]\right]$

$\frac{(l-l)}{k}\cos\frac{n\pi}{l} = 0$, $\frac{h}{(l-a)}\frac{\sin kl}{k^2} = 0$

$= \frac{2}{l}\frac{h \sin ka}{k^2}\left(\frac{1}{a} + \frac{1}{l-a}\right) = \frac{2h}{a(l-a)k^2} \sin ka.$

$= \frac{2hl^2}{a(l-a)n^2\pi^2} \sin \frac{n\pi a}{l}.$

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$$\therefore y(x,t) = \sum \frac{2hl^2}{a(l-a)n^2\pi^2} \sin\frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

$$= \frac{2hl^2}{a(l-a)\pi^2}\left[\sum \frac{1}{n^2} \sin\frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}\right]$$

$$= \frac{2hl^2}{a(l-a)\pi^2}\left[\sin\frac{\pi a}{l} \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} + \frac{1}{4}\sin\frac{2\pi a}{l} \sin\frac{2\pi x}{l} \cos\frac{2\pi ct}{l} + \frac{1}{9}\sin\frac{3\pi a}{l} \sin\frac{3\pi x}{l} \cos\frac{3\pi ct}{l} + \cdots\right]$$

as $a_n \propto \frac{1}{n^2}$

so amplitudes of high freq components fall off rapidly.

If $a = \frac{l}{q}$ where q is an integer,

$\sin\frac{n\pi a}{l} = \sin\frac{n\pi}{q}$. so when $n = pq$ where p is an integer, the term $\sin\frac{n\pi a}{l}$ will vanish.

i.e. disappearance of components for which $n = pq$ ($\therefore n = q, 2q, 3q \cdots$ etc).

If $q = 2$, $a = \frac{l}{2}$ then second, 4th etc i.e. all even harmonics will be absent. ; all these harmonics have a node $a = \frac{l}{q}$.

$\therefore$ The statement that 'In case of a plucked string, harmonics which have a node at a pt of plucking will be all absent.' This is known as Young's law. OR Young-Helmholtz law

Struck ~~String~~ String.

A stretched string which are made to vibrate by striking it with hammer, (e.g. piano) is Struck Str[ing]. There is some velocity at the point of struck in the struck string. The initial condition of the struck string is said to be 'kinetic'.

So the factors — relative masses of hammer and string, the striking velocity of hammer, duration of contact etc.

Assumed :- 1) the duration of contact is very small so that the impact has ceased before the disturbance has had time to spread over an appreciable function of the length.

2) The motion is treated free, with given initial velocity concentrated on a short length.

Let, a stretched string of length l be struck at an infinitesimally small region from $x=r$ to $r+dx$, that this portion acquires an initial velocity u. so the initial condition → the displacement and veloc[ity] are zero everywhere except over the region r to $r+d$[x] where the displacement is zero, but the velocity is [...]

The general expression for the displacement of a stretche[d] string at any point x and time t is given by,

$$y(x,t) = \sum \sin\frac{n\pi x}{l}\left(a_n \cos \omega_n t + b_n \sin \omega_n t\right)$$

putting the initial condition that $y=0$ for all values of [x] at $t=0$

it becomes, $y_0 = y(x,0) = \sum a_n \sin\frac{n\pi x}{l} = 0.$

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Since $\sin \frac{n\pi x}{l}$ ~~[illegible]~~ can't be zero for all values of x

$\therefore a_n = 0$.

~~This gives, $y(x,t) = \sum \omega_n b_n$.~~

$\therefore y(x,t) = \sum b_n \sin \frac{n\pi x}{l} \sin \omega_n t \rightarrow (1)$.

which gives, $\dot{y}(x,t) = \sum \omega_n b_n \sin \frac{n\pi x}{l} \cos \omega_n t \rightarrow (2)$

and $\dot{y}_0 = \dot{y}(x,0) = \sum \omega_n b_n \sin \frac{n\pi x}{l} \rightarrow (3)$.

here y_0 and $\dot{y}_0$ represent respectively the initial displacement and initial velocity at any point x,

Calculation of b_n:–

Multiply both sides of (3) by $\sin \frac{m\pi x}{l}$ where m has some particular value, and integrate for the entire string.

Then $\int_0^l \dot{y}_0 \sin \frac{m\pi x}{l}\, dx = \sum_{n=1}^{\infty} \int_0^l \omega_n b_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx$

when $m \neq n$ the integral will vanish.

Only term that remains is when $m = n$.

$\therefore \int_0^l \dot{y}_0 \sin \frac{m\pi x}{l}\, dx = \omega_n b_n \int_0^l \sin^2 \frac{n\pi x}{l}\, dx = \omega_n b_n \frac{l}{2}$

Now, $\dot{y}_0 = 0$ all over x except from r to $r+dx$ where its value is u.

$\therefore \omega_n b_n \frac{l}{2} = \int_r^{r+dx} u \sin \frac{n\pi x}{l}\, dx = \sin \frac{n\pi r}{l} \int_r^{r+dx} u\, dx$.

$= U \sin \frac{n\pi r}{l}$. where $U = \int_r^{r+dx} u\, dx$

putting $\omega_n = \frac{n\pi c}{l}$ and $\frac{\omega_n l}{2} = \frac{n\pi c}{2}$

$b_n = \frac{2U}{n\pi c} \sin \frac{n\pi r}{l} \rightarrow (4)$ $\therefore b_n \propto \frac{1}{n}$

So greater number of harmonics will have appreciable amplitudes, sound emitted by struck string will be richer in harmonics compared to plucked string.

So, the resultant displacement of the struck string

$$y(x,t) = \frac{2U}{\pi c}\sum \frac{1}{n} \sin\frac{n\pi r}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi c}{l}t.$$

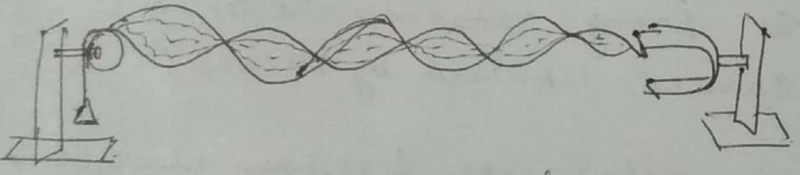
$$= \frac{2U}{\pi c}\left\{ \sin\frac{\pi r}{l} \sin\frac{\pi x}{l} \sin\frac{\pi c t}{l} + \frac{1}{2}\sin\frac{2\pi r}{l}\sin\frac{2\pi x}{l}\sin\frac{2\pi c t}{l} + \dots + \frac{1}{n}\sin\frac{n\pi r}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi c t}{l} + \dots \text{ to } \infty \right\}.$$

If $nr = l$, $\sin\frac{n\pi r}{l} = 0$.

$\therefore$ nth, 2nth etc harmonics will be absent and will have zero amplitude. i.e. all these harmonics have a node at r, the point of striking.

MELDE'S EXPERIMENT

A very good method of demonstrating the resonant vibrations of stretched strings is due to Melde. A silk cord or a very thin light wire is attached to the extremity of one prong of a large tuning fork, preferably electrically



maintained. The string passes over a pulley and carries a scale pan at the other end. When the load (inclusive of weight of scale pan) stretching the string has one of a set of definite values, the string is set to resonant stationary vibration when the fork is excited.

The number of loops in which the string vibrates, depends upon the load and upon the manner of vibration of the fork. The fork may vibrate (1) perpendicular to the length of the string (transverse) (2) parallel to it (longitudinal).

(i) Transverse arrangement:- Let f be the freq of the fork. For resonance with the nth harmonic of the string,

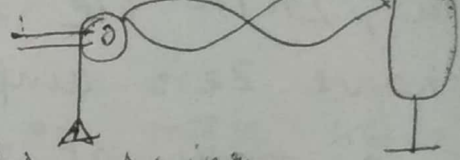
$f = \frac{n}{2l}\sqrt{\frac{T}{m}}$ where l is the length of the string (betn pulley and fork)

$\because$ f, l, and m are const, $\therefore$ $n\sqrt{T}$ = const or, Tn^2 = const.

This means that total load required to make the same string vibrate in 1, 2, 3, ... loops are inversely proportional to the squares of these numbers

If a load of 100 gm makes a string of given length vibrate in one segment, a load of 25 gm will make it vibrate in two segments when the exciting fork is same.

Note :- The fork end of the thread has slight transverse motion, so that the effective length of the thread is slightly diff from actual length.



(ii) Longitudinal arrangement :-

Here the string is tightest when the prong is at the end of it-s swing away from the pulley. It sags most when it is nearest to the pulley.

When the prong. swings away from the position it stretches the string to its maximum when at the other end of the swing, when the prong returns, the string no longer sags, but is carried upward by virtue of its inertia.

When the prong reaches its position nearest to the pulley a second time, the string acquires its maximum displacement above the horizontal. Thus during one complete vibration of the fork, the string executes half a vibration. The frez of the string is therefore half that of the fork.

$$\frac{f}{2} = \frac{n}{2l}\sqrt{\frac{T}{m}}$$

where $= \frac{1}{2l}\sqrt{\frac{T}{m}}$ frez. of string

when the string is vibrating in n loops, As before n^2T is const, but the stretching force now required to produce a particular number of loops is only one fourth compared to that required in transverse vibration.

If no of loops. $f = \frac{p}{l}\sqrt{\frac{T}{m}}$.

ANALYSIS OF COMPLEX VIBRATIONS

SUPERPOSITION OF n HARMONIC WAVES

Fourier Series and Fourier's Theorem:-

If a function $f(x)$ of some real variable x, defined and integrable in the interval $(c, c+2l)$ satisfies Dirichlet's conditions i.e.

i) $f(x)$ is single valued and finite

ii). $f(x)$ is piecewise continuous having at most finite number of finite discontinuities,

iii). $f(x)$ has at most finite number of maxima and minima within the interval,

then $f(x)$ can be expanded in this interval into the following series, called Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right). \rightarrow (1)$$

where a_0, a_n and b_n are constants.

$$\left.\begin{aligned} a_0 &= \frac{1}{l} \int_c^{c+2l} f(x)\, dx. \\ a_n &= \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx \\ b_n &= \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx. \end{aligned}\right\} (2)$$

Evaluation of Fourier Coefficients :-

for a_0.
integrate (1).

$$\int_c^{c+2l} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2l} dx + 0 + 0 .$$ [∵ integrals of sine and cosine terms are zero

or, $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx .$$

for a_n multiply both side of (1) by $\cos\frac{n\pi x}{l}$ and integrate

$$\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx = 0 + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\right)\cdot \cos\frac{n\pi x}{l}\,dx$$

where $\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0$ for all integrals v of m and n.

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \quad \text{for } m \neq n$$

$$= \frac{2l}{2} \quad \text{for } m = n .$$

$$\therefore \int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx = a_n l .$$

or, $$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx .$$

Similarly, $$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx .$$

Fourier's Theorem:- is a set of conditions sufficient to imply the convergence of the Fourier series to some function closely related to the given fn. If $f(x)$ and $f'(x)$ are sectionally continuous in $-l < x < l$ and $f(x)$ satisfies Dirichlet conditions then according to Fourier Theorem.

The series (1) converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ only within the interval.

- At continuous points the series (1) converges to the given fn.
- At discontinuities the series converges to the arithmetic mean of left hand and right hand limiting values
- If $f(x)$ is periodic and comprises a full period $[-l, +l]$ i.e. $f(x) = f(x+2l)$ then the series converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ for all values of x. If $f(-l) \neq f(+l)$

then at the extremum the series converges to

$\frac{1}{2}[f(-l^+) + f(+l^-)]$.

Fourier Series for Periodic vibration:-

If $c = 0$ and $2l = T$. then for periodic motion Fourier series may be written as,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t).$$

where $a_0 = \frac{2}{T} \int_0^T f(t)\, dt.$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega t\, dt.$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega t\, dt.$$

In some cases interval is chosen as, $c = -T/2$ and $c + 2l = +T/2$. then the series is unchanged but the co-efficients becomes;

$$a_0 = \frac{2}{T} \int_{-T/2}^{+T/2} f(t)\, dt.$$

$$a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \cos n\omega t\, dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \sin n\omega t\, dt.$$

Even and Odd functions:-

If $f(t)$ is an even function of time t, i.e. $f(-t) = f(t)$ then for the even limit $[-T/2, +T/2]$, $b_n = 0$ and if $f(t)$ is odd i.e. $f(-t) = -f(t)$ then $a_n = 0$.

Because any odd function integrated over an even limit always yields zero.

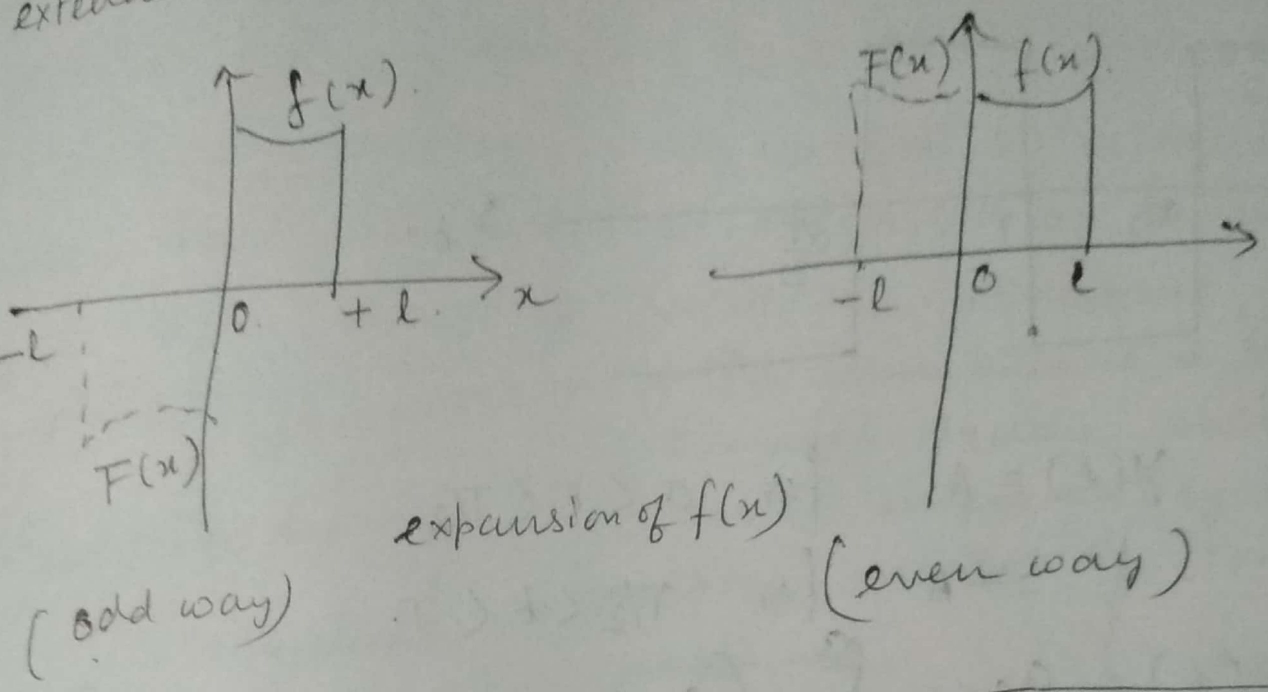
Thus Fourier series for even fn's contains only cosine terms

for odd " " " sine terms

Half range series:-

In some cases it is required to expand a fn. $f(x)$ in the range $(0, l)$ which is half the period of Fourier series.

As the nature of the fn. outside the range $0 < x < l$ is immaterial we can extend the fn. arbitrarily to cover the range $-l < x < +l$ so that the new fn. $F(x)$ may be odd or even. The Fourier series of $F(x)$ ~~coto~~ contains either sine or cosine terms. Within $(0, l)$ the expansion of $F(x)$ coincides with $f(x)$. If it is required to expand $f(x)$ as sine series in $0 < x < l$ then $f(x)$ is extended in odd fashion. To get half range cosine series $f(x)$ is to be extended in even fashion:



expansion of f(x)

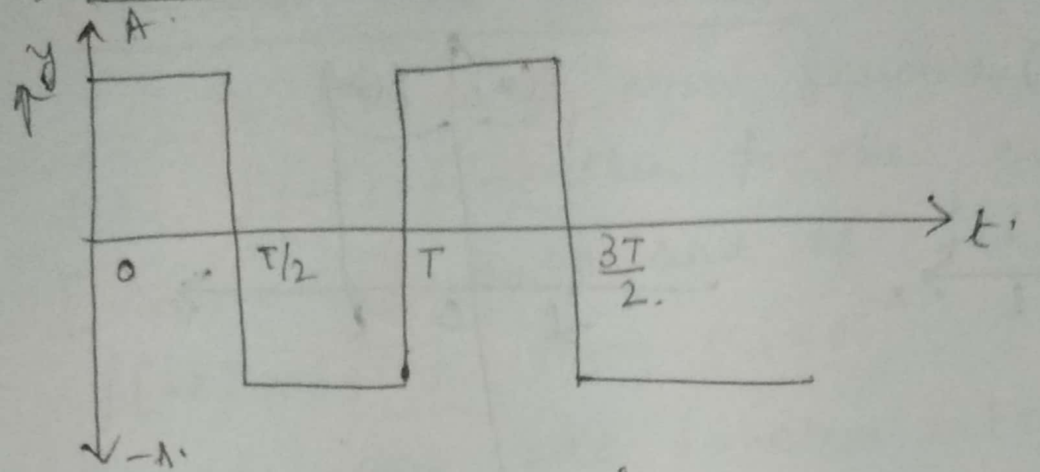
expansibility of a function:-

If a function satisfies ~~&~~ Dirichlet condition then only it can be expanded into a Fourier series.

1) Let $f(x) = \tan x$ in the interval $(-\pi, \pi)$ it can't be expanded into Fourier series as there is infinite discontinuity at $\pm \pi/2$.

1) $f(x) = \frac{1}{x}$ in interval $(0, 4)$ does not satisfy Dirichlet conditions at ~~a~~ $\lim_{x \to 0^+} f(x)$ does not exist

1) SQUARE WAVEFORM:-



$y(t) = A$ for $0 < t < T/2$

$= -A$ for $T/2 < t < T$.

Let, $y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$.

where $\omega = \frac{2\pi}{T}$

Now $a_0 = \frac{2}{T} \int_0^T y(t)\, dt = \frac{2}{T}\left[\int_0^{T/2} A\, dt + \int_{T/2}^{T} (-A)\, dt\right] = 0$

$a_n = \frac{2}{T} \int_0^T y(t) \cos n\omega t\, dt$.

$= \frac{2}{T}\left[\int_0^{T/2} A \cos n\omega t\, dt + \int_{T/2}^{T} (-A) \cos n\omega t\, dt\right]$

$= 0$ [$\because y(t)$ is an odd fn]

$b_n = \frac{2}{T} \int_0^T y(t) \sin n\omega t\, dt$

$= \frac{2}{T}\left[\int_0^{T/2} A \sin \omega t\, dt + \int_{T/2}^{T} (-A) \sin n\omega t\, dt\right]$

$= \frac{2A}{T}\left[-\frac{\cos n\omega t}{n\omega}\Big|_0^{T/2} + \frac{\cos n\omega t}{n\omega}\Big|_{T/2}^{T}\right]$

$= \frac{A}{n\pi}\left[1 - \cos n\pi + \cos 2n\pi - \cos n\pi\right]$

$= \frac{2A}{n\pi}(1 - \cos n\pi) = 0$ if n is even

$= \frac{4A}{n\pi}$ if n is odd.

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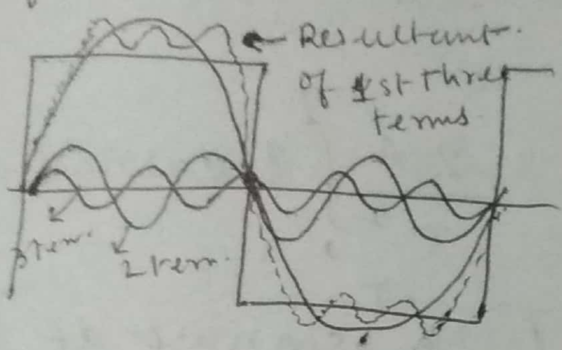
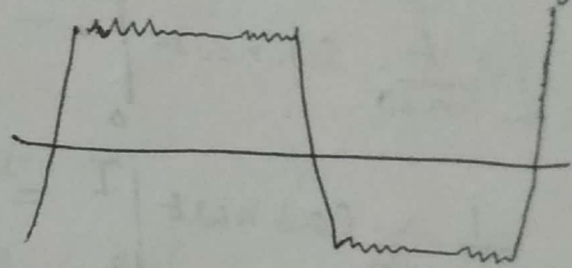
The required Fourier Series becomes :–

$$y(t) = \frac{4A}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right) \rightarrow (A)$$

So a periodic motion represented by a square wave pulse can be considered as formed by the superposition of a large number of harmonic vibrations.



The resemblence to the original square waveform can be improved by considering a large number of terms



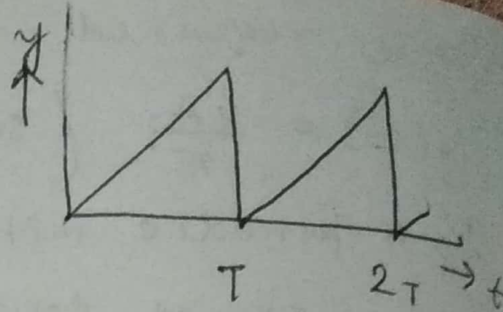
1 st fifteen terms of the series.

Series Convergence:–

Series is convergent and the amplitudes of higher harmonics becomes progressively smaller as the freq increases. Majority of vibrations in acoustics are relatively smooth fns of time and the convergence of Fourier Series in these cases are rapid and only a first few terms gives reasonably good equivalence to the original fn.

2) SAW TOOTH WAVEFORM:—

$$y(t) = \frac{A}{T} \cdot t \quad \text{for } 0 < t < T.$$



$$\therefore a_0 = \frac{2}{T}\int_0^T y(t)\,dt$$

$$= \frac{2}{T}\int_0^T \frac{A}{T}\, t\,dt = \frac{2A}{T^2} \cdot \frac{T^2}{2} = A.$$

$$a_n = \frac{2}{T}\int_0^T y(t)\cos n\omega t\,dt = \frac{2A}{T^2}\int_0^T t\cos n\omega t\,dt$$

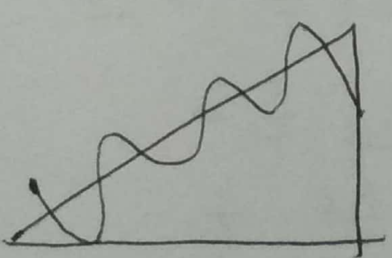
$$= \frac{2A}{T^2}\left[\frac{t}{n\omega}\sin n\omega t\Big|_0^T - \frac{1}{n\omega}\int_0^T \sin n\omega t\,dt\right]$$

$$= \frac{2A}{T^2}\,\frac{1}{n^2\omega^2}\cos n\omega t\Big|_0^T = 0.$$

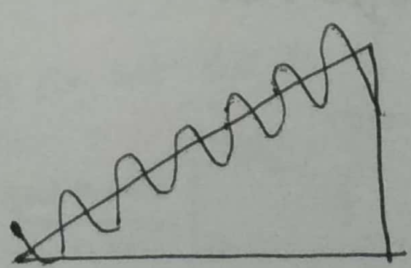
$$b_n = \frac{2}{T}\int_0^T y(t)\sin n\omega t\,dt = \frac{2A}{T^2}\int_0^T t\sin n\omega t\,dt$$

$$= \frac{2A}{T^2}\left[-\frac{t}{n\omega}\cos n\omega t\Big|_0^T + \frac{1}{n\omega}\int_0^T \cos n\omega t\,dt\right]$$

$$= \frac{2A}{T^2}\left[-\frac{T}{n\omega} + \frac{1}{n\omega^2}\sin n\omega t\Big|_0^T\right] = -\frac{A}{n\pi}$$



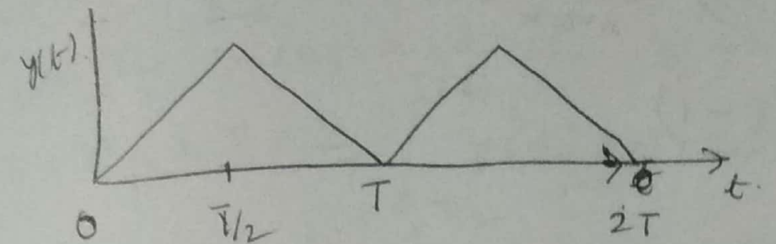
1st three terms



1st 6 terms

$$\therefore y(t) = \frac{A}{2} - \frac{A}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right.$$

(3) TRIANGULAR WAVEFORM :-



$$\frac{y(t)}{t} = \frac{A}{T/2}$$

or, $y(t) = \frac{2A}{T} t$ for $0 < t < \frac{T}{2}$

$$\frac{y(t)}{T-t} = \frac{A}{T/2}$$ or, $y(t) = \frac{2A}{T}(T-t)$ for $\frac{T}{2} < t < T$

So, $a_0 = \frac{2}{T}\int_0^T y(t)\,dt$

$$= \frac{2}{T}\left[\int_0^{T/2} \frac{2A}{T} t\,dt + \int_{T/2}^{T} \frac{2A}{T}(T-t)\,dt\right]$$

$$= \frac{4A}{T^2}\left[\left.\frac{t^2}{2}\right|_0^{T/2} + \left.\left(Tt - \frac{t^2}{2}\right)\right|_{T/2}^{T}\right.$$

$= A.$

$$a_n = \frac{2}{T}\int_0^T y(t)\cos n\omega t\,dt.$$

$$= \frac{2}{T}\left[\int_0^{T/2} \frac{2A}{T} t\cos n\omega t\,dt + \int_{T/2}^{T} \frac{2A}{T}(T-t)\cos n\omega t\,dt\right.$$

$$= \frac{4A}{T^2}\left[\left(\frac{t}{n\omega}\sin n\omega t + \frac{1}{n^2\omega^2}\cos n\omega t\right)\Big|_0^{T/2} + \frac{T}{n\omega}\sin \omega t\Big|_{T/2}^{T} - \left(\frac{t}{n\omega}\sin n\omega t + \frac{1}{n^2\omega^2}\cos n\omega t\right)\Big|_{T/2}^{T}\right]$$

All sine terms give zero contribution.

$$= \frac{4\pi}{T^2}\left[\frac{1}{n^2\omega^2}(\cos n\pi - 1) - \frac{1}{n^2\omega^2}(1-\cos n\pi)\right]$$

$$= \frac{8A}{T^2}\frac{1}{n^2\omega^2}(\cos n\pi - 1)$$

$$= \frac{2A}{\pi^2 n^2}(\cos n\pi - 1)$$

$\therefore a_n = 0$ for n even

$= \frac{-4A}{\pi^2 n^2}$ for n odd.

As the given fn is even, the co-eff b_n must be zero.

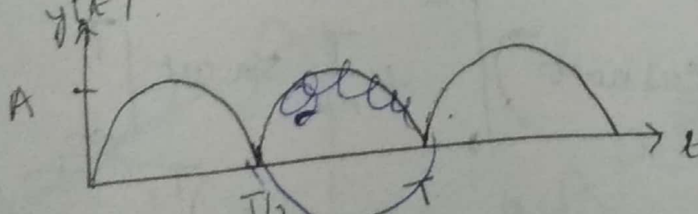
$$b_n = \frac{2}{T}\int_0^T y(t)\sin n\omega t\, dt = 0.$$

$\therefore$ Fourier series becomes;

$$y(t) = \frac{A}{2} - \frac{4A}{\pi^2}\left(\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + \dots\right)$$

(4) FULL WAVE FORM:—

A full waveform of time period T and peak value A is represented by, $y(t) = A\sin\omega t$ for $0<t<T/2$
$= -A\sin\omega t$ for $T/2<t<T$



$$a_0 = \frac{2}{T}\int_0^T y(t)\,dt$$

$$= \frac{2}{T}\left[\int_0^{T/2} A\sin\omega t\, dt - \int_{T/2}^{T} A\sin\omega t\, dt\right]$$

$$= \frac{2A}{T}\left[-\frac{\cos\omega t}{\omega}\Big|_0^{T/2} + \frac{\cos\omega t}{\omega}\Big|_{T/2}^{T}\right] = \frac{2A}{T\omega}(2+2) = \frac{4A}{\pi}$$

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$$a_n = \frac{2}{T}\int_0^T y(t)\cos n\omega t\, dt$$

as the integrand is symmetric about the point $t = \frac{T}{2}$

$$a_n = 2 \times \frac{2}{T}\int_0^{T/2} y(t)\cos n\omega t\, dt.$$

$$= \frac{4A}{T}\int_0^{T/2} \sin\omega t \cos n\omega t\, dt.$$

$$= \frac{2A}{T}\int_0^{T/2} \left[\sin(n+1)\omega t - \sin(n-1)\omega t\right] dt.$$

$$= \frac{2A}{T}\left[-\frac{\cos(n+1)\omega t}{(n+1)\omega} + \frac{\cos(n-1)\omega t}{(n-1)\omega}\right]_0^{T/2}$$

$$= \frac{A}{\pi}\left[-\frac{\cos(n+1)\pi}{(n+1)} + \frac{1}{n+1} + \frac{\cos(n-1)\pi}{(n-1)} - \frac{1}{n-1}\right]$$

$\therefore a_n = 0$ for n odd.

$= -\frac{4A}{\pi(n^2-1)}$ for n even.

As the given fn. $y(t)$ is even so $b_n = 0$.

$$y(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=2,4,6\ldots} \frac{1}{n^2-1}\cos n\omega t$$

$$= \frac{2A}{\pi} - \frac{4A}{\pi}\left[\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \ldots\right]$$

Wave Train and the concept of wave group:-

A monochromatic wave extends from $-\infty$ to $+\infty$. both in space and time. $\psi(x,t) = Re\, A e^{j(\omega t - kx)}$

- But it has no practical importance because no. practical source vibrates indefinitely and its amplitude does not remain const.
- In practice, we get a train of waves of finite duration of ~~of~~ damped amplitude. It is called wave train which is formed by the superposition of a theoretically infinite number of plane harmonic waves having continuously differing frequencies. For practical purposes, the frequencies are limited within a finite range, depending upon the length of the wave train.

The shorter is the length of the wave train, the wider is the effective freq range. So that, a short enough wave train may be considered as a group of harmonic waves which may be called a wave group or wave packet.

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Phase Velocity, Group velocity and their relationship:-

When a monochromatic harmonic wave moves through a medium, the velocity with which the planes of const phase move is called the phase velocity or wave velocity.

A plane harmonic wave is reperented by.

$$\psi(x,t) = \text{Re}\, A e^{j(\omega t - kx)}.$$

for planes of const phase., $\omega t - kx = \text{const}$.

$\therefore$ differentiating, $\frac{d}{dt}(\omega t - kx) = 0$

$$\text{or, } \frac{dx}{dt} = \frac{\omega}{k} = v.$$

where v is the phase velocity.

Group velocity:

Practical waves are of finite duration and are not truly monochromatic. A short enough wave train may be considered as a wave group formed by the superposition of an infinite number of plane harmonic waves having slightly different frequencies and phases. Wave group has a max^m. amplitude and it falls off to zero, not far from max^m.

- If the medium is dispersive, i.e the phase vel depends on freq, then the shape of the group changes, as it travels.
- Velocity of this wave is different from their component wave and is called group velocity.

Relation betn. phase velocity and group velocity

Let us consider a wave group formed by the superposition of a large number of harmonic waves having frequencies in the range $\omega_0 - \frac{\Delta\omega}{2}$ to $\omega_0 + \frac{\Delta\omega}{2}$ where $\Delta\omega \ll \omega_0$.

Let, $$\psi(x,t) = \int_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} A(\omega)\, e^{j(\omega t - kx)}\, d\omega \rightarrow (1)$$

Where $A(\omega)$ is the amplitude of the wave corresponding to the freq ω.

Let the amp of all waves are same i.e.

$A(\omega) = A_0$ within the chosen spectr

and zero outside.

For a dispersive medium, the wave number k depends on ω i.e. $k = k(\omega)$.

Since the freq lies within a small interval k can be expanded as,

$$k = k(\omega_0) + (\omega - \omega_0)\frac{dk}{d\omega_0} + \cdots$$

Substituting into (1).

$$\psi(x,t) = A_0\, e^{j(\omega_0 t - k(\omega_0)x)} \int_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} e^{j(\omega - \omega_0)\left[t - \left(\frac{dk}{d\omega}\right)_0 x\right]} d\omega \rightarrow (2)$$

$$= A_0\, e^{j\left[\omega t - \left[k(\omega_0) + (\omega - \omega_0)\frac{dk}{d\omega_0}\right]x\right]} d\omega = A_0\, e^{j\left[\omega_0 t + (\omega t - \omega_0 t) - \left[k(\omega_0) + (\omega - \omega_0)\frac{dk}{d\omega_0}\right]\right]} d\omega.$$

Let $\omega' = \omega - \omega_0$.

and $t - \left(\frac{dk}{d\omega}\right)_0 x = m$.

$$\therefore \psi(x,t) = A_0 e^{j(\omega_0 t - k_0 x)} \int_{-\frac{\Delta\omega}{2}}^{+\frac{\Delta\omega}{2}} e^{j\omega' m} \, d\omega'.$$

$$= A_0 e^{j(\omega_0 t - k_0 x)} \cdot \left. \frac{e^{jm\omega'}}{jm} \right|_{-\Delta\omega/2}^{+\Delta\omega/2}.$$

$$= \frac{A_0 e^{j(\omega_0 t - k_0 x)}}{jm} \cdot \left(e^{jm\frac{\Delta\omega}{2}} + e^{-jm\frac{\Delta\omega}{2}} \right)$$

$$= \frac{2 \sin m\Delta\omega/2}{m} A_0 e^{j(\omega_0 t - k_0 x)}$$

$$= \frac{2 \sin\left[\left\{t - \left(\frac{dk}{d\omega}\right)_0 x\right\} \frac{\Delta\omega}{2}\right]}{t - \left(\frac{dk}{d\omega}\right)_0 x} A e^{j(\omega_0 t - k_0 x)} \longrightarrow (3)$$

which represents a wave whose amplitude is modulated by a factor

$$\frac{2 \sin\left[\left\{t - \left(\frac{dk}{d\omega}\right)_0 x\right\} \frac{\Delta\omega}{2}\right]}{t - \left(\frac{dk}{d\omega}\right)_0 x}.$$

which has a principal maxima when

$$t - \left(\frac{dk}{d\omega}\right)_0 x = 0$$

$\therefore$ The group maximum is not situate at a fixed pt but moves in space with a velocity, called group velocity,

$$v_g = \frac{dx}{dt} = \frac{d\omega}{dk}$$

$d\omega$

Since phase velocity $v = \frac{\omega}{k}$ or $\omega = vk$

$$\therefore v_g = \frac{d}{dk}(vk) = v + k\frac{dv}{dk} = v + k\frac{dv}{d\lambda}\frac{d\lambda}{dk}$$

Now $k = \frac{2\pi}{\lambda}$ or, $\frac{d\lambda}{dk} = \frac{-2\pi}{k^2}$

$$\therefore v_g = v - \frac{2\pi}{k}\frac{dv}{d\lambda} = v - \lambda\frac{dv}{d\lambda}.$$

$$\therefore \boxed{v_g = v - \lambda\frac{dv}{d\lambda}}$$ for a dispersive medium

In a nondispersive medium $\frac{dv}{d\lambda} = 0$

$$\therefore \boxed{v_g = v}$$ for non dispersive medium.

- Sound waves in audible range donot show dispersion in any medium but ultrasonic waves show dispersion.

Importance of Group velocity:

It is used to carry energy or information where a single monochromatic wave can't send a signal because such wave can't be changed. So in order to send a signal / message some properties of the wave will be modulated at the sending end such that it can be decoded at the receiving end. This is called modulated wave, which is a mixture of group of harmonic waves. This type of wave travels with group velocity.

- The energy of the signal thus travel with group ~~velocity~~ and has the velocity of group.
