Dept. of Mathematics

2nd Semester

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Sets in $\mathbf{R}$

Topics: Limit point, Bolzano-Weierstrass theorem, Boundary point

Limit point

Let S be a subset of $\mathbb{R}$. A point p in $\mathbb{R}$ is said to be a limit point (or an accumulation point, or a cluster point) of S if every neighbourhood of p contains a point of S other than p.

Therefore p is a limit point of S if for each positive ε, $[N(p, \varepsilon) \sim \{p\}] \cap S \neq \phi$.

$N(p, \varepsilon) \sim \{p\}$ is called the deleted ε-neighbourhood of p and is denoted by $N'(p, \varepsilon)$. $N(p) \sim \{p\}$ is called the deleted neighbourhood of p and is denoted by $N'(p)$.

. Therefore p is a limit point of S if every deleted neighbourhood of p contains a point of S.

Note: A limit point of S may or may not belong to S.

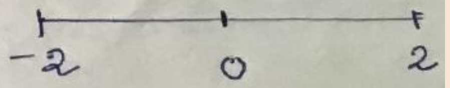
Example

To show that 0 is a limit point of $[-2\ 2)$

Sol^n

Aim:

$$\boxed{\forall\ \varepsilon > 0,\ N'(0, \varepsilon) \cap [-2, 2) \neq \phi}$$



Take any $\varepsilon > 0$, and choose

$$y = \max\{-2, -\tfrac{\varepsilon}{2}\}$$

then $y \in N'(0, \varepsilon)$ and $y \in [-2, 2)$

so $N'(0, \varepsilon) \cap [-2, 2) \neq \phi$.

Isolated point

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an isolated point of S if x is not a limit point of S.

Since x is not a limit point of S, $\exists$ a neighbourhood $N(x)$ of x such that $N'(x) \cap S = \phi$. Since $x \in S$, $N(x) \cap S = \{x\}$

therefore x is an isolated point of S if for some positive ε, $N(x, \varepsilon)$ contains no point of S other than x.

Example

To show that 3 is an isolated point of $\{1, 2, 3, 4\}$

Sol^n

Aim:

$$\boxed{\exists\ \varepsilon > 0 \text{ s.t } N'(3, \varepsilon) \cap \{1, 2, 3, 4\} = \phi}$$

Choose $\varepsilon = \frac{1}{2}$

then $N'(3, \varepsilon) = \left[\left(3 - \frac{1}{2}, 3 + \frac{1}{2}\right) \setminus \{3\}\right] \cap \{1, 2, 3, 4\} = \phi$

$\Rightarrow$ 3 is an isolated point of $\{1, 2, 3, 4\}$.

Theorem

Let $S \subseteq \mathbb{R}$ and P be a limit point of S. Then every neighbourhood of P contains infinitely many elements of S.

Proof

Aim:

$\boxed{\forall\, \varepsilon > 0 \; N(P, \varepsilon) \text{ has infinitely many points of } S.}$

By definition of limit point

$\forall\, \varepsilon > 0 \quad N'(P, \varepsilon) \cap S \neq \phi$

Let $B = N'(P, \varepsilon) \cap S$. We prove that B is an infinite set.

Suppose B is not an infinite set

$\Rightarrow$ B contain only a finite number of elements of S, say $\alpha_1, \alpha_2, \dots, \alpha_m$

Let $\delta_i = |P - \alpha_i|$, $i = 1, 2, \dots, m$, $\Rightarrow \delta_i > 0$ $\forall i = 1, 2, \dots, m$

Let $\delta = \min\limits_i \delta_i$

$\Rightarrow \delta > 0$ and $\alpha_i \notin N(P, \delta)$, $i = 1, 2, \dots, m$

$\Rightarrow N'(P, \delta) \cap S = \phi$

which contradicts the fact that P is a limit point of S

$\Rightarrow$ B is an infinite set

$\Rightarrow$ $N(P, \varepsilon)$ contains infinitely many elements of S, for each $\varepsilon > 0$. (Proved)

Theorem

Let $S \subseteq \mathbb{R}$. Then every interior point of S is a limit point of S.

Proof

Let x be an interior point of S.

$\Rightarrow \exists\ \delta > 0$ s.t. $N(x, \delta) \subseteq S$.

Aim:

$\boxed{\forall\ \varepsilon > 0 \quad N'(x, \varepsilon) \cap S \neq \phi}$

Case I

$0 < \varepsilon < \delta$

$\Rightarrow N(x, \varepsilon) \subset N(x, \delta) \subseteq S$

$\Rightarrow N'(x, \varepsilon) \cap S \neq \phi$.

Case II

$\varepsilon \geq \delta$

$\Rightarrow N(x, \delta) \subseteq N(x, \varepsilon)$

Since $N(x, \delta) \subseteq S$ and $N(x, \delta) \subseteq N(x, \varepsilon)$

$\Rightarrow N(x, \delta) \subseteq N(x, \varepsilon) \cap S$.

$\Rightarrow N'(x, \varepsilon) \cap S \neq \phi$.

In both the cases $N'(x, \varepsilon) \cap S \neq \phi$

$\Rightarrow x$ is a limit point of S. (proved)

Theorem (Bolzano-Weierstrass theorem)
Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof

Let S be a bounded infinite subset of $\mathbb{R}$
$\Rightarrow$ $\sup S$ and $\inf S$ both exist.
$\Rightarrow$ $s_* \le x \le s^* \ \forall x \in S$, where $s_* = \inf S$
$s^* = \sup S$.

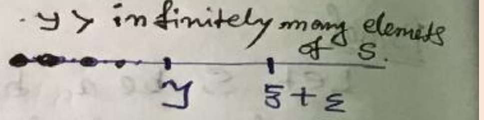
Let H be a subset of $\mathbb{R}$ defined by
$H = \{ x \in \mathbb{R} : x$ is greater than infinitely many elements of $S \}$
$\Rightarrow$ $s^* \in H$ and so H is a non-empty subset of $\mathbb{R}$.
Since $h \in H \Rightarrow h$ is greater than infinitely many elements of S
$\Rightarrow$ $h > s_*$, because $s_* \le x \ \forall x \in S$, so no element less or equal to s_* belongs to H.
$\Rightarrow$ H is a non-empty subset of $\mathbb{R}$, which is bounded below, s_* being a lower bound.
$\Rightarrow$ $\inf H$ exists. Let $\inf H = \xi$

Aim:
ξ is a limit point of S i.e $\forall \varepsilon > 0 \ N'(\xi, \varepsilon) \cap S \neq \phi$.

Let $\varepsilon > 0$

Since $\xi = \inf H$, so by the definition of infimum

$\exists\, y \in H$ s.t $\xi \leq y < \xi + \varepsilon$.

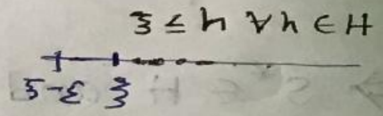


Since $y \in H$

$\Rightarrow$ y is greater than infinitely many elements of S.

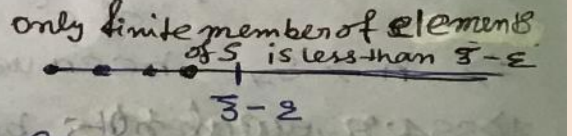
$\Rightarrow$ $\xi + \varepsilon$ is greater than infinitely many elements of S.

Again, since ξ is the infimum of H



$\Rightarrow$ there doesn't exist any $h \in H$ s.t $h < \xi$

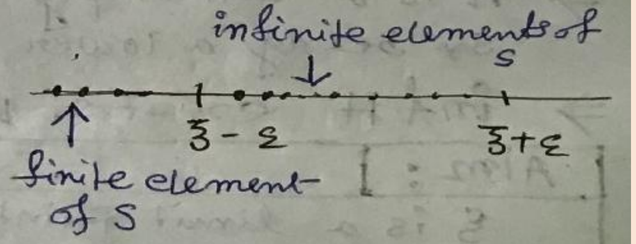
Since $\xi - \varepsilon < \xi$ as $\varepsilon > 0$



$\Rightarrow$ $\xi - \varepsilon \notin H$.

$\Rightarrow$ $\xi - \varepsilon$ can exceed at most a finite number of elements of S.

$\Rightarrow$ $(\xi - \varepsilon, \xi + \varepsilon)$ contains infinitely many elements of S.



$\Rightarrow$ $N'(\xi, \varepsilon) \cap S \neq \phi$

Since ε is arbitrary. so ξ is a limit point of S.

(Proved)

Example

verify the Bolzano-Weierstrass theorem for the set $\{1+\frac{1}{n} : n \in \mathbb{N}\}$

Sol^n

Let $A = \{1+\frac{1}{n} : n \in \mathbb{N}\}$

$\Rightarrow$ A is infinite, because $1+\frac{1}{n}$ is distinct and $\mathbb{N}$ is infinite.

Also A is bounded, Because $\forall n \in \mathbb{N}$ $1 < 1+\frac{1}{n} \leq 2$

So, by Bolzano-Weierstrass theorem, A must have a limit point in $\mathbb{R}$.

No, we shall show that 1 is a limit point of A.

Aim:

$$\forall \varepsilon > 0 \quad N'(1, \varepsilon) \cap A \neq \phi$$

Take any $\varepsilon > 0$.

Then By Archimedean property,

$\exists n \in \mathbb{N}$ s.t $0 < \frac{1}{n} < \varepsilon$

$\Rightarrow 1+\frac{1}{n} \in (1, 1+\varepsilon) \subseteq N'(1, \varepsilon)$

$\therefore N'(1, \varepsilon) \cap A \neq \phi$

Since $\varepsilon > 0$ is any arbitrary number

$\Rightarrow$ 1 is a limit point of the set A

Boundary point

Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a boundary point of S if every neighbourhood $N(x)$ of x contains a point of S and also a point of $\mathbb{R} - S$ (i.e S^c)

$\boxed{\text{i.e } \forall\, \varepsilon > 0 \quad N(x, \varepsilon) \cap S \neq \phi \text{ and } N(x, \varepsilon) \cap S^c \neq \phi}$

Example

To show 1 is a boundary point of $[0, 1)$.

Sol^n

$\boxed{\text{Aim:}}$

$\boxed{\forall\, \varepsilon > 0 \quad N(1, \varepsilon) \cap [0, 1) \neq \phi \text{ and } N(1, \varepsilon) \cap [0, 1)^c \neq \phi}$

First we shall show $N(1, \varepsilon) \cap [0, 1) \neq \phi$

i.e $\exists\, y \in N(1, \varepsilon) \cap [0, 1)$

choose $y = \max\{0, 1 - \varepsilon/2\}$

then $y \in N(1, \varepsilon)$ and $y \in [0, 1)$

$\Rightarrow N(1, \varepsilon) \cap [0, 1) \neq \phi$

Next, we shall show that $N(1, \varepsilon) \cap [0, 1)^c \neq \phi$

i.e $\exists\, z \in N(1, \varepsilon) \cap [0, 1)^c$

Choose $z = 1$

$\Rightarrow z \in N(1, \varepsilon)$ But $z \notin [0, 1) \Rightarrow z \in [0, 1)^c$

so $N(1, \varepsilon) \cap [0, 1)^c \neq \phi$

$\Rightarrow$ 1 is a boundary point of $[0, 1)$

Example:

Let $A \subseteq \mathbb{R}$ and $l \notin A$. Prove that l is a limit point of A if and only if it is a boundary point of A.

Sol^n

$\boxed{l \notin A.}$ given.

Let l is a limit point of A

$\Rightarrow \forall \ \varepsilon > 0 \quad N'(l, \varepsilon) \cap A \neq \phi$ ...... ①

Again for all $\varepsilon > 0 \quad l \in N(l, \varepsilon) \cap A^c$ ... ②

From ① and ②

$\Rightarrow$ For all $\varepsilon > 0 \quad N(l, \varepsilon) \cap A \neq \phi$ and $N(l, \varepsilon) \cap A^c \neq \phi$

$\Rightarrow l$ is a boundary point of A

Conversely, l is a boundary point of A

$\Rightarrow \forall \ \varepsilon > 0 \quad N(l, \varepsilon) \cap A \neq \phi$ and $N(l, \varepsilon) \cap A^c \neq \phi$

Since $l \notin A$ and for $\forall \varepsilon > 0 \ N(l, \varepsilon) \cap A \neq \phi$

$\Rightarrow \forall \ \varepsilon > 0 \quad N'(l, \varepsilon) \cap A \neq \phi$

$\Rightarrow l$ is a limit point of the set A.

Practice Problems

1. Give an example of an infinite set $S \subseteq \mathbb{R}$ such that
   (i) S has no limit point
   (ii) S has three limit points

2. Let $S = \{(-1)^n (1 + \frac{1}{n}) : n \in \mathbb{N}\}$.
   Show that -1 and 1 are limit points of S.

3. Let $S = \{\frac{(-1)^m}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$
   (i) Show that 0 is a limit point of S.
   (ii) If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of S.
   (iii) If $k \in \mathbb{N}$, show that $\frac{-1}{2k-1}$ is a limit point of S.

4. Verify Bolzano-Weierstrass theorem for the set $S \subseteq \mathbb{R}$
   (i) $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$
   (ii) $S = \{\frac{n-1}{n+1} : n \in \mathbb{N}\}$

5. Give an example of a boundary point that is not a limit point, and a limit point that is not a boundary point.

6. Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Show that a is either a limit point of A or a limit point of A^c (or both).