3

SOUND

3.1 Free Vibration

We know that a body can vibrate only if it is in *stable* equilibrium position. For such a body, restoring force develops when it is displaced from its equilibrium position. Simplest vibration is simple harmonic motion; here restoring force (F) is proportional to displacement (x) from the equilibrium position.

By free vibration we mean a vibration where there is *no frictional force opposing* the vibration. We know that in absence of frictional force, the equation of motion for free simple harmonic vibration of a particle of mass m is

$$F = m\frac{d^2x}{dt^2} = -kx$$

$$\text{or,}\quad \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x \quad \ldots\ldots\ldots\ldots\ldots\ 3.1$$

Here k is *force constant* of the system, which is the restoring force developed per unit displacement from mean position. It is constant for a particular system.

The solution of the above differential equation is given by

$$x = a\cos(\omega t + \delta) \quad \ldots\ldots\ldots\ldots\ldots\ldots 3.2$$

From this equation we get the displacement (x) at any time (t). Once the body is displaced and then left to itself, it starts vibrating with *constant* amplitude a and *angular frequency* (ω). As there is *no loss of energy* from the body, it continues its vibration *forever* according to the above equation. Its total energy is given by

$$E = \frac{1}{2}m\omega^2 a^2 = \text{constant.} \quad \ldots\ldots\ldots\ldots 3.3$$

The time-displacement curve of the motion is shown in Fig.3.1. You notice the x-t curve touches the two parallel lines $x = +a$ and $x = -a$ periodically, because the amplitude remains constant.

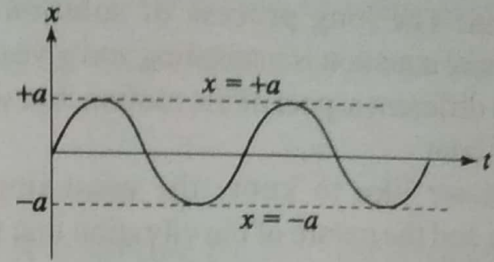


Fig. 3.1

The frequency of this vibration is given by

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots 3.4$$

This frequency is called the *natural frequency* of vibration, which is *characteristic* of the vibrating system.

Obviously free vibration is an *ideal motion*; we never find it in macro-world (the world of big bodies). In the micro-world, i.e., where individual atoms or molecules are involved, such motions are possible.

3.2 Damped simple harmonic motions

In all *real* vibrations different kinds of frictional forces come into play. All these forces together are called the *damping force*. The vibration that takes place in presence of damping forces is called *damped vibration*.

Generally this damping force (F_d) is proportional to the velocity (v) of the vibrating body. Such a system is said to be *linearly* damped.

$$\therefore F_d = Dv$$

D is a constant. D is retarding force per unit velocity.

$\therefore$ Equation of a damped simple harmonic vibration is given by

$$m\frac{d^2x}{dt^2} = -kx - D\frac{dx}{dt} \quad \ldots\ldots\ldots\ldots\ldots 3.5$$

The negative sign before the damping force implies that it is *opposing* the motion.

Damping force per unit mass is given by

$$\frac{F_d}{m} = \frac{D}{m}v = 2b\frac{dx}{dt} \quad \ldots\ldots\ldots\ldots 3.6$$

The constant 'b' is called the *damping factor*, its dimension is $[T^{-1}]$. Larger is damping, bigger is b (half of damping force per unit mass).

Using the above notations we can write the eqn. 3.5 as

$$\frac{d^2x}{dt^2} = -\omega^2 x - 2b\frac{dx}{dt} \quad \ldots\ldots 3.7$$

To know the detailed nature of the damped vibration, we have to solve this differential equation. The long process of solution of the differential equation is interesting and gives insight into the different aspect of the motion, but we shall solve it later.

First we like to know the most important solution and the nature of the vibration that follows from this solution.

When *damping is small*, $b < \omega$, solution of eq. 3.7 is found to be

$$x = ae^{-bt}\cos\left(\sqrt{\omega^2 - b^2}\,t + \delta\right) \quad \ldots\ldots 3.8$$

Let us compare this equation with the eqn. of s. h. m.

$x = a\cos(\omega t + \delta)$

We can easily recognise that it represents an oscillatory motion.

The initial phase δ depends upon the initial condition, value of displacement when we start our description.

For example, if at $t = 0, x = a$, i.e. the particle is at the extreme position when we start observation, then $\cos\delta = 1$, $\therefore \delta = 0$.

The equation of the damped oscillatory motion is $x = ae^{-bt}\cos\left(\sqrt{\omega^2 - b^2}\,t\right)$ 3.8a

$\therefore$ The amplitude of damped vibration at time t is $a_t = a_0 e^{-bt}$.

Angular frequency, $\omega_d = \sqrt{\omega^2 - b^2}$ and

Frequency, $n_d = \dfrac{\sqrt{\omega^2 - b^2}}{2\pi}$.

If, on the other hand, the particle is at the mean position when we start observation, then at $t = 0$, $x = 0$. $\therefore \cos\delta = 0$ and $\delta = \pi/2$. The equation of the damped oscillatory motion is

$$x = ae^{-bt}\sin\left(\sqrt{\omega^2 - b^2}\,t\right) \quad \ldots\ldots 3.8b$$

Thus we find that when the damping force is *small*, it has two effects on the vibration :

(1) The frequency of vibration is *reduced* somewhat. The new reduced frequency is given by

$$n_d = \frac{\sqrt{\omega^2 - b^2}}{2\pi} < n \quad \ldots\ldots 3.9$$

It is called the *damped* frequency. Greater the damping force, lesser is the damped frequency.

Since the damping force, in general, is not very large, n_d is *nearly equal* to the natural frequency.

(2) The amplitude of vibration slowly decreases with time. In fact it decreases *exponentially* with time according to the equation

$$a_t = a_0 e^{-bt} \quad \ldots\ldots 3.10$$

Hence a_0 is the *initial* amplitude (at $t = 0$) and a_t is the amplitude at time t after starting of the vibration.

Such a vibration is called *damped simple harmonic vibration.*

The time-displacement curve for such a vibration is shown in Fig 3.2. We notice how the amplitude of vibration decreases with time. The x-t curve periodically touches the two exponentially decreasing curves :

$x = \pm a_0 e^{-bt}$.

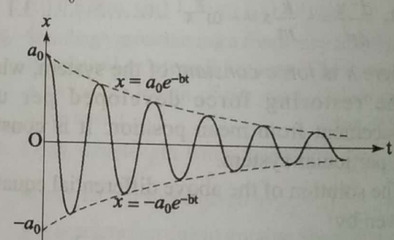


Fig. 3.2

The motion, strictly speaking, is not periodic, because it never comes back to where it starts from. But for small damping it can be regarded as periodic.

If we put $b = 0$, in the above eqns. n_d becomes equal to natural frequency n and a_t becomes equal to the initial amplitude a_0. Thus *damping forces are responsible* for the above effects.

There are two interesting situations.

(1) *Critical damping* : Now suppose that the damping is so large that $\omega = b$, then $n_d = 0$. In this condition the body *does not oscillate* but returns to the equilibrium position without oscillation when it is slightly displaced and then released. This condition is called *critical damping.*

$\therefore$ In critical damping condition $b = \sqrt{\frac{k}{m}}$.

(2) *Overdamping* : If the damping is still bigger ($b > \omega$), the system returns to its equilibrium position *more slowly* than with critical damping. This condition is called overdamping.

We can see in detail,how all these types of motion arise in the solution of the differential equation given below.

3.2.1 *Logarithmic decrement* :

When a particle undergoes damped vibration its amplitude decreases gradually, as we have seen above. Now we like to whether there is a definite pattern in that process.

From equation of damped vibration $x = ae^{-bt}$ $\cos\omega_d t$, we find that maximum displacement on the two sides will occur when $\cos\omega_d t = \pm 1$, i.e., at time $t = \frac{m\pi}{\omega_d}$, where $m = 0, 1, 2, ...$

$\therefore$ Successive maximum displacements on the two sides are given by

$$x_1 = a,\ x_2 = -ae^{-\frac{\pi b}{\omega_d}},\ x_3 = ae^{-\frac{2\pi b}{\omega_d}},$$

$$x_4 = -ae^{-\frac{3\pi b}{\omega_d}}, \text{ so on.}$$

Ignoring the signs, we find that maximum displacements on the two sides decrease gradually obeying a definite rule :

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = ... = \frac{x_{n-1}}{x_n} = e^{\frac{\pi b}{\omega_d}} = e^{\lambda}, \text{ say. ...(i)}$$

The quantity λ is called the logarithmic decrement, because λ can be written as $\lambda = \log_e\left(\frac{x_1}{x_2}\right)$

Also from (i) we can write

$$\frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times ..... \times \frac{x_{n-1}}{x_n} = \frac{x_1}{x_n} = e^{(n-1)\lambda}$$

or $(n-1)\lambda = \log_e\left(\frac{x_1}{x_n}\right)$

Finding $x_1$and x_2 or x_1 and x_n, we can find λ.

Again, $\lambda = \frac{\pi b}{\omega_d} = \frac{\pi b}{2\pi n_d} = \frac{bT_d}{2}$ where T_d is the time period of the damped vibration.

Therefore knowing λ and T_d we can find the damping factor b.

Now let us suppose that a particle starts from the mean position. From eqn. 3.8b we can say that if there is no damping force, the particle would have gone to the extreme position $x = a$. But as damping force is there, the first maximum displacement x_1 will be somewhat smaller than a.We like to calculate the effect of the damping. It reaches the extreme position at time $t = \frac{T_d}{4} = \frac{2\pi}{4\omega_d} = \frac{\pi}{2\omega_d}$.

Displacement, $x_1 = ae^{-b\frac{\pi}{2\omega_d}} \sin\frac{\pi}{2} = ae^{-\frac{\lambda}{2}}$

Hence observing the first displacement x_1, we can take correction for damping. We have $a = x_1 e^{\frac{\lambda}{2}} = x_1\left(1+\frac{\lambda}{2}\right)$, for small damping when λ is small.

Such corrections are very important for ballistic galvanometer where the first throw gives the amount of charge passed through it.

3.2.2 *Loss of energy due to damping forces :*

When damping is small we can take $\omega_d \approx \omega$ and put the value of amplitude from eqn. 3.10 in eqn. 3.3, we get *approximate* value of the total energy.

$\therefore$ Total energy of a damped oscillator,

$$E(t) = \frac{1}{2}m\omega^2 a_0^2 e^{-2bt}$$

$$= E_0 e^{-2bt} \text{ ....3.11}$$

Here E_0 is the initial energy. Eqn. 3.11 shows that the system loses energy exponentially at a *higher rate* than displacement x. Owing to continuous loss of energy in moving against the damping force F_d, energy of vibration decreases and ultimately the body stops after sometime.

In time $\tau = \frac{1}{2b}$, energy reduces to $E = \frac{E_0}{e} \approx 0.367E_0 = 36.7\%$ of initial energy. This time τ is called the decay time or *time constant* of the damped system.

Now we calculate the energy loss in duration of *one time period*. It is given by

$\Delta E = E_0 - E = E_0(1-e^{-2bT}) = E_0\ (1-1+2bT)$

$= E_0 2bT$

As damping is small we keep the first two terms in the exponential series.

A damped oscillator is often described by its Q factor (*quality factor*). It is a measure of the rate of loss of its energy owing to damping. It is defined as $Q = \omega/2b$.

$$\therefore \frac{\Delta E}{E_0} = 2bT = \frac{\omega}{Q}T = \frac{2\pi}{Q}$$

We see that *fractional energy loss per cycle* is inversely proportional to Q. So, *bigger is Q lesser* is the energy loss per cycle. And Q is bigger when damping factor b is smaller. That means that a system with *lesser damping has higher quality factor*.

We have seen that a body capable of vibrating has *two* characteristic frequencies: its *natural* frequency, n and its *damped* frequency, n_d. If the body is disturbed and then left to itself, it always vibrates with n_d, because the damping force acts. If the damping force could be reduced to zero, only then the body could vibrate with its natural frequency. But still the natural frequency n has importance, as we shall see below, when resonance occurs.

Solution of the differential equation :

$$\frac{d^2x}{dt^2} = -\omega^2 x - 2b\frac{dx}{dt}.$$

We take the *trial* solution: $x = f(t)e^{-bt}$ and see whether and in which conditions it satisfies this solution.

We have $\frac{dx}{dt} = \left(\frac{df}{dt} - bf\right)e^{-bt}$,

$$\therefore \frac{d^2x}{dt^2} = \left(\frac{d^2f}{dt^2} - 2b\frac{df}{dt} + b^2 f\right)e^{-bt}$$

Substituting these in the above equation we get,

$$\left(\frac{d^2f}{dt^2} - 2b\frac{df}{dt} + b^2 f\right)e^{-bt}$$

$$= -\omega^2 f e^{-bt} - 2b\left(\frac{df}{dt} - bf\right)e^{-bt}$$

$$\therefore \frac{d^2f}{dt^2} + \left(\omega^2 - b^2\right)f = 0 \quad \text{.............(i)}$$

Thus we find that the function f must satisfy the above differential equation.

There can *be three different situations* depending upon the *relative* values of ω and b.

Case I : If $b > \omega$, i.e., the *damping is large*, the above eqn. can be written as

$$\frac{d^2f}{dt^2} - \left(b^2 - \omega^2\right)f = 0 \quad \text{..................(ii)}$$

We take the trial solution of this equation as $f = e^{\beta t}$. To find β, we substitute it in the equation.

We have $\frac{d^2f}{dt^2} = \beta^2 e^{\beta t}$.

We get $\beta^2 e^{\beta t} - (b^2 - \omega^2)e^{\beta t} = 0$. $\therefore \beta^2 = (b^2 - \omega^2)$.

$\therefore$ There are *two* possible values of β given by

$$\beta = \pm\sqrt{b^2 - \omega^2}$$

In this case, the required solution of the damped system will be the *linear combination* of these two solutions. We have

$$x = e^{-bt}\left[Ae^{\sqrt{b^2-\omega^2}\,t} + Be^{-\sqrt{b^2-\omega^2}\,t}\right]$$

Here A and B are arbitrary constants, whose values can be determined from the *initial* conditions (displacement and velocity) of the system.

We know exponential function either increases or decreases monotonically, it does not oscillate. Therefore if the system is displaced and then released, *there is no oscillation*. Damping factor b is large, therefore owing to term e^{-bt}, x decreases exponentially to zero. The body comes back to its initial position ($x=0$) *asymptotically*. Asymptotically means that x is really zero after a very large time. Such a system is often called *over-damped, aperiodic or dead-beat.*

Case II : If $b = \omega$, the system is called *critically damped.*

In this condition, the eqn. (ii) for f becomes

$$\frac{d^2f}{dt^2} = 0.$$

Integrating this we get $\frac{df}{dt} = A$(constant).

Integrating again we get, $f = A + Bt$, where B is another constant.

$\therefore$ The motion of the *critically damped system* is governed by the equation $x = e^{-bt}(A+Bt)$.

In this case *also* once the system is displaced and then left to itself; it comes back to its rest position without any oscillation. But in this case it comes back *quicker* than in the previous case.

In both these two situations, the system is called *aperiodic*.

We often try to obtain the condition of critical damping in a system, when we want the system to avoid oscillations and return to equilibrium position quickly. For this purpose, shock absorbers are used to damp the oscillations of a car on its springs. Similarly sensitive galvanometers in physics laboratories are often critically damped, so that the spot of light returns to its rest position without unnecessary oscillations.

In Fig. 3.3, we can see how displacement (x) of the particle decreases with time (t) in overdamped and critically damped conditions.

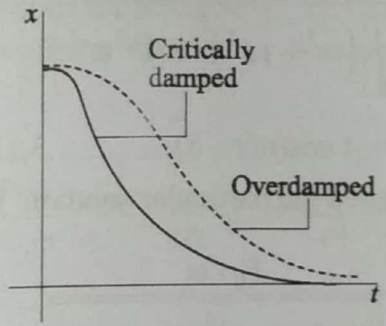


Fig. 3.3

Case III : If $b < \omega$, i.e., the *damping is small*, the equation for f becomes $\frac{d^2 f}{dt^2} = -(\omega^2 - b^2)f$

This is the basic equation for simple harmonic motion, with ω^2 replaced by $\omega^2 - b^2$. So the solutions are the corresponding sine and cosine functions.

∴ The general solutions of the above equation is

$$f = A\cos\left(\sqrt{\omega^2 - b^2}\,t\right) + B\sin\left(\sqrt{\omega^2 - b^2}\,t\right) \text{ ....(iii)}$$

Let us put $A = a\cos\delta$ and $B = a\sin\delta$. Then the eqn. (iii) can be written in a compact form :

$f = a\cos\left(\sqrt{\omega^2 - b^2}\,t - \delta\right)$, where a and δ are given by $a = \sqrt{A^2 + B^2}$ and $\tan\delta = \frac{B}{A}$.

Putting this f in the eqn. $x = f(t)e^{-bt}$, we get the equation of *oscillatory motion* of a *lightly* damped system as given by

$$x = ae^{-bt}\cos\left(\sqrt{\omega^2 - b^2}\,t - \delta\right)$$

Thus we get the derivation of all the different kind of motion of a damped system, as discussed above]

3.3 Forced Vibration

Suppose there is a body *capable of vibrating* and having natural frequency n. and damped frequency n_d. We like to study what happens when a *periodic* external force *begins* to act on it.

The force is *not constant*. Both magnitude and direction of the force F change *sinusoidally* with time according to the equation:

$$F = F_0 \cos\omega' t \text{ ....3.12}$$

Here F_0 is the amplitude of the applied force and is its angular frequency. The frequency (n') and time-period (T') of the *applied force* is given by

$$n' = \frac{1}{T'} = \frac{\omega'}{2\pi} \text{ ....3.13}$$

Now let us first describe *qualitatively* what happens if such a force begins to act on a body, capable of vibrating.

When the force just begins to act, the body *starts its vibration* with *two frequencies*:

(i) its damped frequency, n_d and (ii) the frequency n' of the applied force.

As a result the vibration will be a little erratic; you cannot find a definite period. But the vibration with the damped frequency n_d will soon be damped out because of continuous loss of energy owing to the damping force, as we have discussed above. This initial state of vibration is called transient state of vibration.

The body will then continue its vibration with the frequency n' of the *external* force as long as the force acts. This is called the *steady state* of vibration. In fact the energy loss against the damping force is supplied by the system exerting the external force. Hence the body can vibrate with that frequency as long as the force acts.

In the steady state, the body is made to vibrate with the frequency of the applied force. Hence it is called forced vibration.

The amplitude of such a forced vibration is generally *small*. We find that the *amplitude of forced vibration depends* upon the *frequency* of the applied force and on the damping factor.

Therefore a body, capable of vibrating, can be made to vibrate with *any frequency*, if a force of that frequency is applied to it.

As amplitude of forced vibration depends on the frequency of the *forcing* system,a remarkable phenomenon occurs. This is called *resonance.*

If the *frequency* of driving force is *exactly equal to the natural frequency* of the body, the body begins its vibration with *large amplitude* and the *velocity or energy* of vibration has the *maximum value.* This special kind of forced vibration is called velocity or energy *resonance.* In this condition the vibrating system takes in maximum energy from the forcing system.

Now we shall see how all the above conclusions and much more follow from the differential equations for a system under the action of a driving simple harmonic force.

3.3.1 *Differential equation of motion of forced vibration and its solution:*

First let us write down the equation of motion for forced vibration.

There are three forces acting on the system during its motion:

(i) restoring force $-kx$, (ii) damping force $-D\frac{dx}{dt}$ and (iii) an external periodic force $F = F_0 \cos\omega' t$.

∴ Equation of motion for the forced vibration is

$$m\frac{d^2x}{dt^2} = -kx - D\frac{dx}{dt} + F$$

Positive sign before the force F implies that the force *aids* the motion. The equation is written in a compact form as

$$\frac{d^2x}{dt^2} = -\omega^2 x - 2b\frac{dx}{dt} + \frac{F_0}{m}\cos\omega' t$$

or, $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\cos\omega' t \quad \text{.........3.14}$$

We have put

$$2b = D/m \text{ and } f = \frac{F_0}{m} \quad \text{..................3.15}$$

Also we know that the natural angular frequency $\omega = \sqrt{\frac{k}{m}}$.

This is a non-homogeneous second order differential equation. We shall solve it latter. First we assume the solution and discuss what follows from the solution.

The general solution (x) of eqn. 3.14 has two terms : $x = x_1 + x_2$

(i) One term (x_1) is the solution of the above eqn. 3.14, with the R.H.S. equal to *zero.* We have already seen the solution of this homogeneous equation. It is the eqn. 3.8.

$$x_1 = ae^{-bt}\cos\left(\sqrt{\omega^2 - b^2}\,t - \delta\right) \quad \text{................3.16}$$

It is called the complementary function.

(ii) The other term (x_2) is of the form

$$x_2 = \frac{F_0/m}{\sqrt{\left\{\left(\omega^2 - \omega'^2\right)^2 + 4b^2\omega'^2\right\}}}\cos(\omega' t - \delta)$$

$$= A\cos(\omega' t - \delta), \quad \text{...........3.17}$$

It is called the particular solution. We have pu[t]

$$A = \frac{F_0/m}{\sqrt{\left\{\left(\omega^2 - \omega'^2\right)^2 + 4b^2\omega'^2\right\}}} \quad \text{.........3.18a}$$

Here the initial phase difference is δ given b[y]

$$\tan\delta = \frac{2b\omega'}{\left(\omega^2 - \omega'^2\right)} \quad \text{.........................3.18b}$$

∴ Solution of the differential eqn.3.14 is

$$x = ae^{-bt}\cos\left(\sqrt{\omega^2 - b^2}\,t - \delta\right) + A\cos\left(\omega' t - \delta\right) \quad \text{.................3.1}$$

Discussions :

Existence of two terms in the solution impli[es] that the body *begins* its vibration with *two angul[ar] frequencies*:

(i) the damped frequency of the syste[m]

$n_d = \frac{\sqrt{\omega^2 - b^2}}{2\pi}$ and

(ii) the frequency of the applied force $n' = \frac{\omega'}{2\pi}$

Owing to the presence of the factor e^{-bt}, t[he] amplitude of the vibration with the damp[ed] frequency n_d diminishes *exponentially with tim[e]* as we have seen above.

Therefore after sometime, the motion due to x_1 disappears, the body continues vibrating with constant amplitude A and with the frequency n' according to eqn. 3.17, as long as the driving force acts. This is the *steady forced vibration*. The solution x_2 represents the steady state solution.

The behaviour of the system while it is approaching the steady state is called the '*transient*' response. The transient response depends on all the different factors; in particular, the time taken by the system to reach the steady state depends on the properties of the system (b) and on the initial conditions.

In Fig.3.4, we see the curves describing vibrations corresponding to x_1, x_2 and x. We see that the resultant vibration x is irregular in the

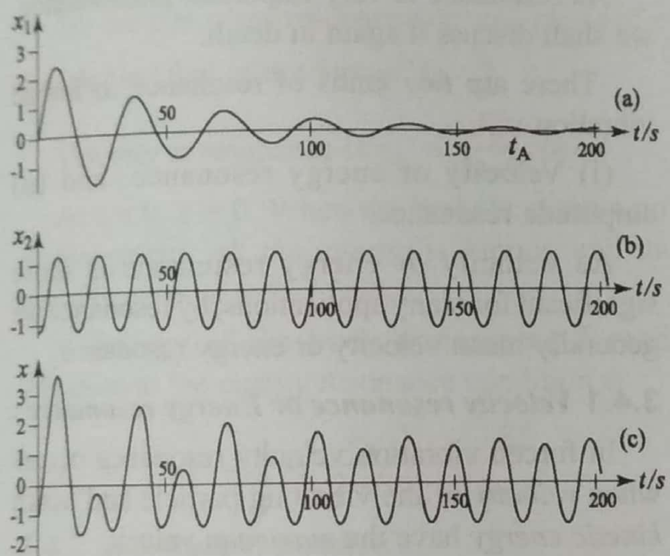


Fig. 3.4

initial unsteady or transient state until x_1 ceases and then steady forced vibration x_2 starts.

Looking at the solution, eqn.3.17, we observe the following:

(i) There is a phase difference (δ) between the applied periodic force and the displacement x_2 the displacement x_2 *lags behind* the force by angle δ. Phase difference depends on b, ω and ω'.

(ii) Amplitude (A) of forced vibration depends F_0,m,ω as is expected, but in addition it also depends on the angular frequency ω' of the driving force and on the damping factor b.

If ω' is *much different* from the natural frequency ω of the body, the amplitude of forced vibration is *very small*, as $(\omega^2 - \omega'^2)^2$ occurs in the denominator of amplitude.

As ω' approaches ω, the amplitude increases.

Also we see that with *increase* of the damping force or damping factor (b),the amplitude of the forced vibration *decreases* and vice versa.

Resonance :

Resonance occurs if the *frequency (n') of the external periodic force is exactly equal to the natural frequency (n) of the body.*

Substituting $\omega = \omega'$ in eqn.3.18a, we see that amplitude of the forced vibration is *very large*, given by

$$A_{res} = \frac{F_0}{2mb\omega'} = \frac{F_0/m}{2b\omega} \qquad \text{.................3.20}$$

From the very beginning the body starts vibrating with very large amplitude and *the energy of vibration has the maximum value*. This special forced vibration is called *resonance*.

It should be noted that in this condition amplitude of the displacement has *not* the maximum possible value, but *velocity* and *kinetic energy* have the *maximum* values. Hence sometimes this condition is called *velocity resonance* or *energy resonance*. Resonance is very important in many applications, as we shall see below.

In this condition the body draws maximum energy from the driving system which applies the periodic force.

Substituting $\omega = \omega'$ in eqn. 3.18b, we find $\tan\delta = \frac{2b\omega'}{0}$ $\therefore \delta = \frac{\pi}{2}$. The vibration at resonance has a phase-difference of $\frac{\pi}{2}$ with the applied force.

We also see from eqn. 3.18a that smaller is the damping factor; greater is the amplitude of vibration at resonance.

Solution of the differential equation of forced vibration **:**

Now we shall solve the differential equation to get the particular solution 3.17.

The differential eqn. is written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\cos\omega' t \qquad \text{..............3.14}$$

To get its solution in a simple way, we construct an equation :

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f\sin\omega' t \qquad \text{............(i)}$$

We multiple both sides of eqn. (i) by the $i = \sqrt{-1}$ and add it to eqn. 3.14. We get

$$\frac{d^2x}{dt^2} + i\frac{d^2y}{dt^2} + 2b\frac{dx}{dt} + i2b\frac{dy}{dt} + \omega^2 x + i\omega^2 y = f\cos\omega' t + if\sin\omega' t$$

Or, $$\frac{d^2}{dt^2}(x+iy) + 2b\frac{d}{dt}(x+iy) + \omega^2(x+iy) = f(\cos\omega' t + i\sin\omega' t)$$

We put $x + iy = z$ and we shall use the well known relation $(\cos\omega' t + i\sin\omega' t) = e^{i\omega' t}$. The above eqn. can, therefore, be written as

$$\frac{d^2z}{dt^2} + 2b\frac{dz}{dt} + \omega^2 z = fe^{i\omega' t} \quad \text{.....................(ii)}$$

This equation can be easily solved for z and once z is found, we can easily find x, as x is the *real part* of z. We like to get the steady state solution for the vibration with frequency ω'. So, we take the trial solution as $z = z_0 e^{i\omega' t}$.

$\therefore \frac{dz}{dt} = i\omega' z$ and $\frac{d^2z}{dt^2} = -\omega'^2 z$. Substituting these in eqn.(ii), we get

$$-\omega'^2 z + i2b\omega' z + \omega^2 z = f\frac{z}{z_0}$$

$$\therefore z_0 = \frac{f}{(\omega^2 - \omega'^2) + i2b\omega'}$$

Now we express the denominator in the polar form.

We put $(\omega^2 - \omega'^2) = a\cos\delta$ and $2b\omega' = a\sin\delta$. Then the denominator becomes $a(\cos\delta + i\sin\delta) = ae^{i\delta}$,

where $a = \sqrt{a^2\cos^2\delta + a^2\sin^2\delta}$

$$= \sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2} \quad \text{........(iii)}$$

Using these results, we can write

$$z_0 = \frac{f}{ae^{i\delta}} = \frac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}e^{-i\delta}$$

$$\therefore z = z_0 e^{i\omega' t} = \frac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}e^{i(\omega' t - \delta)}$$

$$= \frac{f}{a}e^{i(\omega' t - \delta)}$$

As we have seen above x = Real part of z.

Real part of $e^{i(\omega' t - \delta)}$ is $\cos(\omega' t - \delta)$.

$\therefore$ We can write down the particular solution of eqn. 3.14 as

$$x = \text{Re } z = \frac{f}{a}\cos(\omega' t - \delta)$$

$$= \frac{F_0/m}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}\cos(\omega' t - \delta)$$

$$= A\cos(\omega' t - \delta) \quad \text{...............(iv)}$$

Thus we get the desired solution, as mentioned above in eqn.3.17.

3.4 Resonance

As resonance is very important phenomenon, we shall discuss it again in detail.

There are *two* kinds of resonance in forced vibration :

(i) Velocity or energy resonance and (ii) amplitude resonance.

As velocity or energy resonance is more significant for many applications, by resonance we generally mean velocity or energy resonance.

3.4.1 *Velocity resonance or Energy resonance* :

In forced vibration velocity resonance occurs when *velocity* of the vibrating particle and hence *kinetic energy* have the *maximum* values.

Displacement of forced vibration in *steady state* at any instant is

$$x = \frac{F_0/m}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}\cos(\omega' t - \delta)$$

$\therefore$ Velocity, $v = \frac{dx}{dt}$

$$= \frac{-\omega' F_0/m}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}\sin(\omega' t - \delta)$$

Since the ω' is the only variable, velocity v has maximum amplitude when $\omega = \omega'$

Hence the *condition of velocity or energy resonance is* $\omega = \omega'$.

We have mentioned this condition of resonance above.

In resonance condition, $a\cos\delta = (\omega^2 - \omega'^2) = 0$.

$\delta = \frac{\pi}{2}$.

If $\omega > \omega'$, $\cos\delta = +$ve. $\therefore 0 < \delta < \frac{\pi}{2}$

If $\omega < \omega'$, $\cos\delta = -$ ve. $\therefore \frac{\pi}{2} < \delta < \pi$

$\therefore$ At resonance, forced vibration is described by the equation :

$$x = \frac{F_0/m}{2b\omega}\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$= \frac{F_0}{2mb\omega}\sin\omega t = A_{res}\sin\omega t \quad \text{.............3.21}$$

A_{res} is the amplitude at resonance, given by

$$A_{res} = \frac{F_0}{2mb\omega} \quad \text{........................3.21a}$$

We see that when resonance occurs, phase of x *lags behind* that of the force F by $\frac{\pi}{2}$.

Velocity at resonance is $v_{res} = \frac{F_0}{2mb}\cos\omega t$.

At $t = 0$, $x = 0$. When the body is at the zero displacement, *all the energy* is kinetic and its velocity is equal to $F_0 / 2mb$.

$\therefore$ Energy of the body undergoing forced vibration at the energy resonance condition is

$$E_{res} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{F_0}{2mb}\right)^2 = \frac{F_0^2}{8mb^2}.$$

3.4.2 *Amplitude resonance :*

Now let us see in which condition the *amplitude* of vibration has the *maximum* value. Amplitude of forced vibration is

$$A = \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega'^2\right)^2 + 4b^2\omega'^2}}$$

We have to find the condition at which A has maximum value. This corresponds to the frequency ω' for which the *denominator has the minimum value*, i.e.,

$$\frac{d}{d\omega'}[(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2] = 0$$

or, $2(\omega^2 - \omega'^2).(-2\omega') + 4b^2.2\omega' = 0$

or, $\omega^2 - \omega'^2 = 2b^2$.

$$\therefore \omega' = \sqrt{\omega^2 - 2b^2} \quad \text{......................3.22}$$

For this value of ω', *amplitude of forced vibration has the maximum value.*

Hence this condition is called the *amplitude resonance*.

The maximum value of the amplitude is

$$A_{max} = \frac{F_0/m}{\sqrt{4b^4 + 4b^2\left(\omega^2 - 2b^2\right)}}$$

$$= \frac{F_0/m}{\sqrt{4b^2\left(\omega^2 - b^2\right)}} = \frac{F_0/m}{2b\sqrt{\omega^2 - b^2}} \quad \text{......3.23}$$

We find that lesser is damping, larger is A_{max} and in absence of damping ($b = 0$), A_{max} is infinitely large and $\omega' = \omega$.

Comparing eqns. 3.21a and 3.23, we see that $A_{max} > A_{res}$.

Also we find that amplitude resonance occurs at a frequency *lesser* than *both* damped frequency $\left(\sqrt{\omega^2 - b^2}\right)$ and natural frequency (ω) of the system. But for small damping, the difference is not large. Of these two resonances, *velocity resonance is more important*, because in this condition, kinetic energy has maximum value and so *the response of the system is more pronounced.* By resonance we generally mean velocity resonance.

3.4.3 *Power in forced vibration and resonance*

Now we shall calculate the power *supplied by the forcing system* during the forced vibration of a body.

Instantaneous power, $P =$ force (F) $\times$ velocity (v).

$$\therefore P = -F_0\cos\omega' t \times \frac{f\omega'}{a}\sin(\omega' t - \delta),$$

$$= -\frac{F_0^2\omega'}{ma}\cos\omega' t\sin(\omega' t - \delta).$$

where we have put the denominator equal to a. $\therefore a = \sqrt{\left(\omega^2 - \omega'^2\right)^2 + 4b^2\omega'^2}$

$$P = -\frac{F_0^2\omega'}{ma}\left(\frac{1}{2}\sin 2\omega' t\cos\delta - \cos^2\omega' t\sin\delta\right)$$

Now let us find the average power over a period T.

Average value of $\sin 2\omega' t$ and $\cos^2\omega t$ over a period are

$$\frac{1}{T}\int_0^T \sin 2\omega' t\,dt = 0 \text{ and } \frac{1}{T}\int_0^T \cos^2 \omega' t\,dt = \frac{1}{2}$$

$\therefore$ Average power, $P_{av} = \frac{1}{2}\frac{F_0^2\omega'}{ma}\sin\delta$.

At resonance, $\omega = \omega'$, $a = 2b\omega'$ and $\delta = \frac{\pi}{2}$.

$\therefore$ Average energy delivered from the forcing system *at resonance* per second is $P_{res} = \frac{F_0^2}{4mb}$.

This is the maximum power delivered.

3.4.4 *Sharpness of resonance* :

By *sharpness* of resonance we mean the *quality* of resonance. We have seen above that as the forced frequency ω' approaches the *natural* frequency ω, kinetic energy of the forced vibration *increases*, i.e., *the response of the system increases.* Kinetic energy has the maximum value when $\omega' = \omega$.

Now the question is *how rapidly* the response *diminishes* when ω'departs from ω. If a slight difference between ω' and ω, makes the amplitude very small, we say the resonance is *very sharp*. If on the other hand, the amplitude does not decrease noticeably even for a large difference between ω' and ω, we say the resonance is *flat*.

Naturally when the resonance is very sharp, the body or the system is very *selective*. It responds very *strongly* only at a particular frequency or at frequencies very near to a particular frequency; but its response is very small at other frequencies. Such systems are desirable in many occasions. Also it is found that resonance is *sharp when damping is small* and it is flat when damping is large.

In Fig. 3.5, we see amplitude of forced vibration as a *function of* (ω'/ω) for three different damping.

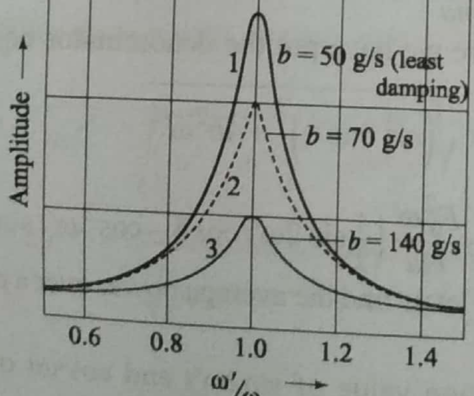


Fig. 3.5

In the figure you notice how amplitude increases as $\omega' \to \omega$ and how the response increases as damping factor b decreases.

We have got a qualitative idea about sharpness of resonance. Now we shall get a *quantitative* measure of the sharpness of resonance.

Natural frequency of the system is ω. So ω is the *resonant frequency*, where the average power of forced vibration has the maximum value P_{res}. Now suppose that at frequencies ω_1 and ω_2 $(\omega_1 < \omega < \omega_2)$ the power reduces to $\frac{1}{2}P_{res}$, i.e., exactly to half its maximum value, see Fig.3.6.

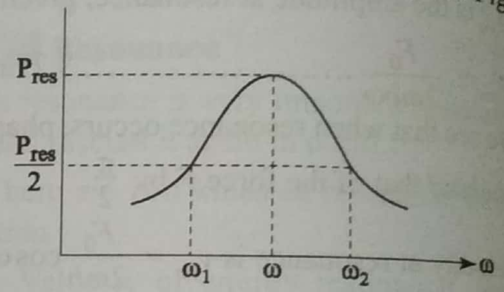


Fig. 3.6

The two frequencies ω_1 and ω_2 are called *half-power frequencies* and the difference $(\omega_2 - \omega_1)$ is called the *band-width.*

Sharpness of resonance is measured by a quantity called *quality factor Q*, defined as

$$Q = \frac{\omega}{\omega_2 - \omega_1} \quad \text{.................3.24}$$

If band-width is small, a slight deviation of the frequency from the resonant frequency (ω) produces a sharp drop in response. Therefore, lesser is the band-width $(\omega_2 - \omega_1)$, larger is the quality factor and the sharpness of resonance, see the figure.

It is quite expected that quality factor Q of a vibrating system should depend on the conditions that *characterise* the system. To find such dependence, let us find the values of ω' for which the power of the vibrating system is exactly equal to half the maximum value, i.e.,

$$P_{av} = \frac{1}{2}P_{res} \quad \text{........................(i)}$$

We have, $P_{av} = \frac{1}{2}\frac{F_0^2\omega'}{ma}\sin\delta$,

$P_{res} = \frac{F_0^2}{4mb}$ and $\sin\delta = \frac{2b\omega'}{a}$

$$\therefore P_{av} = \frac{1}{2}\frac{F_0^2\omega'}{ma}\frac{2b\omega'}{a} = \frac{F_0^2\omega'^2 b}{ma^2}$$

The above condition (i) would be satisfied if

$$\frac{F_0^2\omega'^2 b}{ma^2} = \frac{1}{2}\frac{F_0^2}{4mb}.$$

or, $a^2 = 8b^2\omega'^2$

or, $(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2 = 8b^2\omega'^2$

or, $(\omega^2 - \omega'^2)^2 = 4b^2\omega'^2$

or, $\omega^2 - \omega'^2 = \pm 2b\omega'$

or, $\omega'^2 \pm 2b\omega' - \omega^2 = 0$

Solutions of the equation are given by

$$\omega' = \frac{\mp 2b \pm \sqrt{4b^2 + 4\omega^2}}{2} = \mp b \pm \sqrt{b^2 + \omega^2}$$

Since frequency must be positive, we must choose the two positive solutions only.

$\therefore$ ω_1 and ω_2 are given by $\omega_1 = -b + \sqrt{b^2 + \omega^2}$

and $\omega_2 = b + \sqrt{b^2 + \omega^2}$.

$\therefore \omega_2 - \omega_1 = 2b$

$\therefore$ Quality factor, $Q = \dfrac{\omega}{\omega_2 - \omega_1} = \dfrac{\omega}{2b} = \dfrac{1}{2b}\sqrt{\dfrac{k}{m}}$.

We see that Q is decided by the *damping factor* (b), *force constant* (k) and *mass* (m) of the vibrating system.

Product of the half-power frequencies is given by $\omega_1\omega_2 = b^2 + \omega^2 - b^2 = \omega^2$.

3.4.5 *Examples of forced vibration and resonance :*

Forced vibration and resonance are very important phenomena, useful and interesting, Let us see some examples.

(A) Mechanical Example

We shall first describe the Barton's experiment, which gives a very good illustration of forced vibration and resonance.

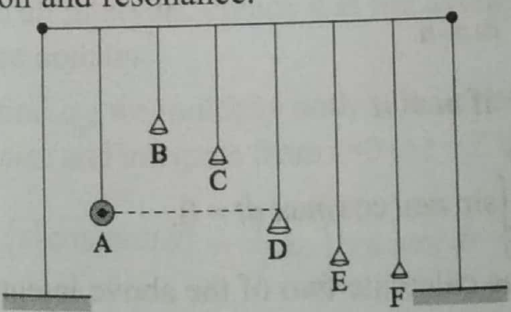


Fig. 3.7

In Fig.3.7, A is heavy metal ball suspended from a thick rubber cord. B, C, D, E and F are also pendulum of different lengths having very light bobs; these may be paper cones. Now A is set into vibration in a direction perpendicular to length of the cord. It is observed that all the pendulums, except D, begin their vibrations erratically and ultimately continue vibrating with the time period equal to that of A. But the amplitudes of their vibrations are *small*. Hence these are examples of forced vibrations as we described above. But the vibration of D, whose length is kept exactly equal to the length of A behaves differently. It picks up the vibration almost immediately and its amplitude grows to a fairly large value. Hence resonance occurs for D.

A very common example of resonance is a child enjoying a swing. If the periodic impulses applied to the swing agrees with the natural frequency of the swing, large amplitude can be attained. Resonance occurs.

If you look around you will see many examples of resonance. The various parts motor car, such as a flexibly mounted engine, brake rods, gear lever etc., have their own natural frequency of vibration. The periodic motion of the pistons communicates to them a forcing frequency proportional to the speed of the car. As the speed of the car alters, the frequency of the pistons may match the natural frequency of some part so that it is thrown into resonant vibration and rattles vigorously.

Resonant vibration is of considerable practical importance to structural and mechanical engineers. If quite a small periodic force operates on some structure or machine having the same natural frequency, vibrations of large magnitude develop. Vibrations produce stresses and, for large magnitudes, the resultant stresses may exceed the elastic limit and damage the structure. Hence in designing a structure it is necessary to examine what external periodic forces may act on the structure. The structure is then built up so as to have a different natural frequency. Great care is taken to make sure that none of the natural frequencies in which a wing of an aircraft can oscillate match the frequency of the engine in flight. In earth quake seismic waves produce vibrations and those structures are completely destroyed, whose natural frequencies match the frequencies of those waves.

(B) Acoustical Example

When a vibrating tuning fork is held in the hand a feeble sound is heard. But if we press its stem on a table-top the sound is greatly magnified. In fact the table is now thrown into forced vibration with the frequency of the fork. A large mass of air in contact with the table is now set into vibration and as a result the volume of the sound increases. But now the rate of loss of energy from the fork is high and so it stops vibrating after a short time.

In string instruments like sitar, esraj, guitar, etc., strings are stretched on a thin wooden board. The vibration of string produces forced vibration of the board and thence of air. This intensifies the emitted sound.

Many of these instruments have several strings tuned to different notes. When a tune is sounded on the principal wire, resonant vibrations are excited in the strings tuned to the same note. This increases both the intensity and the pleasantness of the tune played. In 'percussion' instruments, like drums, the intensity of the sound is increased by the forced vibration of the air inside them.

Loudspeakers of pure quality sometimes give a magnified response to certain parts of the musical scale because of such resonance. The result is bloomy reproduction.

A glass with low damping can be broken by an intense sound wave at a frequency equal to or very nearly equal to the natural frequency of vibration of the glass.

(C) Electromagnetic Example

Reception of radio signal is brought about by resonance. The frequency of the receiver set is adjusted to that of the radio wave coming from a particular station. Hence only that wave produces large response in the receiver.

3.5 Fourier Analysis

We shall first state Fourier's theorem and apply it two problems and then we shall discuss its deep significance.

Statement: Any *single-valued, periodic* function, which is *continuous* or has *a finite number* of *finite* discontinuities in a period, may be expressed as a sum of simple harmonic terms, having frequencies which are *multiples* of the *frequency of the given function*.

Let $y = f(t)$ be a periodic function of time. Its time period is T and angular frequency is $\omega = \frac{2\pi}{T}$. $\therefore f(t) = f(t+T)$. According to Fourier's theorem we can write

$$y = f(t) = \frac{1}{2}a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + ... + b_1 \sin\omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + ...$$

$$\therefore\ y = f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \quad ....3.25$$

Thus the function is expanded into an *infinite series* involving *sines* and *cosines*. The *minimum* frequency is the *frequency of the given periodic function* and others are *integral* multiples of that minimum frequency. The minimum frequency is called *fundamental* frequency and others are harmonics.

Here a's and b's are constants, called Fourier's coefficients, whose values depend on *nature* of the given periodic function. We are to find their values.

Physically the theorem implies that any periodic motion can be analysed into a large number of simple harmonic motions of appropriate frequencies (ω_n) and amplitudes (a_n, b_n) and conversely any periodic function can be synthesized by *adding*(superposing) those simple harmonic components.

The process of decomposition of a periodic function into its simple harmonic components is called *Fourier analysis*.

To find the values of the Fourier coefficients we have to use the following three integrals:

(1) $\int_0^T \sin n\omega t\, dt = \int_0^T \cos n\omega t\, dt = 0$

(2) $\int_0^T \cos n\omega t \cos m\omega t\, dt = \int_0^T \sin n\omega t \sin m\omega t\, dt$

$= 0$, if $m \neq n$

$= \frac{T}{2}$ if $m = n$

(3) $\int_0^T \sin n\omega t \cos m\omega t\, dt = 0.$

Let us calculate two of the above integrals.

$$\int_0^T \sin n\omega t\, dt = -\left[\frac{\cos n\omega t}{n\omega}\right]_0^T$$

$$= -\frac{1}{n\omega}[\cos n\omega T - \cos 0]$$

$$= -\frac{1}{n\omega}[\cos 2n\pi - \cos 0] = 0$$

$$\int_0^T \cos n\omega t \cos m\omega t\, dt$$

$$= \frac{1}{2}\left[\int_0^T \cos(n+m)\omega t\, dt + \int_0^T \cos(n-m)\,\omega t\, dt\right]$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)\omega T - \sin 0}{(n+m)\omega} + \frac{\sin(n-m)\omega T - \sin 0}{(n-m)\omega}\right]$$

$$= 0$$

When $n = m$, the integral becomes

$$\int_0^T \cos^2 n\omega t dt = \frac{1}{2}\int_0^T (1 + \cos 2n\omega t)\, dt$$

$$= \frac{1}{2}\int_0^T dt + \int_0^T \cos 2n\omega t\, dt = \frac{T}{2}.$$

Now let us see how the above Fourier coefficients a's and b's are calculated.

To find a_0, we multiply both sides of eqn.3.25 by dt and integrate from t=0 to t =T. In the right hand side two integrals of cosine and sine terms are zero. We have

$$\therefore \int_0^T f(t)\, dt$$

$$= \frac{1}{2}a_0\int_0^T dt + \sum_{n=1}^{\infty}\left[a_n\int_0^T \cos n\omega t\, dt + b_n\int_0^T \sin n\omega t\, dt\right].$$

$$= \frac{T}{2}a_0.$$

$$\therefore a_0 = \frac{2}{T}\int_0^T f(t)\, dt \quad ....3.26$$

Note that the first constant term is nothing but the *mean value* of the function in the interval [0,T]. Since the function is periodic this average is the same in all intervals. Hence it is the average over its entire domain.

To find a_n, we multiply both sides of eqn.3.25 by cos $m\omega t$ and integrate from t=0 to t =T.We have

$$\int_0^T f(t)\cos m\omega t\, dt = \frac{1}{2}a_0\int_0^T \cos n\omega t\, dt +$$

$$\sum_{n=1}^{\infty}\left[a_n\int_0^T \cos n\omega t \cos m\omega t\, dt + b_n\int_0^T \sin n\omega t \cos m\omega t\, dt\right]$$

$$= \frac{T}{2}a_m$$

Only if n is equal m the integral $\int_0^T \cos m\omega t \cos m\omega t\, dt = \frac{T}{2}$. All other terms are zero.

$$\therefore a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\, dt\,, n = 1,2,3,.... \quad ....3.27$$

To find b_n, we multiply both sides of eqn. 3.25 by sin $m\omega t$ and integrate from t=0 to t =T. We have

$$\int_0^T f(t)\sin m\omega t\, dt = \frac{1}{2}a_0\int_0^T \sin m\omega t\, dt +$$

$$\sum_{n=1}^{\infty}\left[a_n\int_0^T \cos n\omega t \sin m\omega t\, dt + b_n\int_0^T \sin n\omega t \sin m\omega t\, dt\right]$$

$$= \frac{T}{2}b_m$$

Again only one term is non-zero, when

$$n = m,\ \int_0^T \sin m\omega t \sin m\omega t\, dt = \frac{T}{2}$$

$$\therefore b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt,\ \ n = 1,2,3.......3.28$$

Now let us apply this theorem to two periodic functions and express them as the sum of sine and cosine terms.

(1) Square Wave

In Fig.3.8, we see a square wave, which can be described as function $f(t)$ which satisfies the conditions :

$f(t) = +\, a$ from t = 0 to t =T/2.

$f(t) = -\, a$ from t=T/2 to t = T.

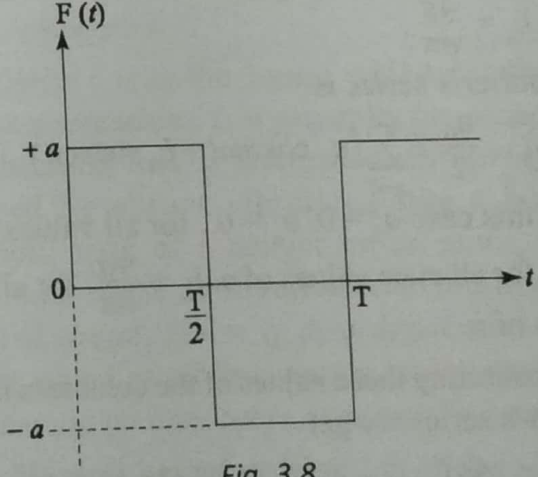


Fig. 3.8

The function is periodic and the time period T and it has finite discontinuities.

Applying Fourier's theorem, we shall nc express this function as a Fourier series of sum sine and cosine terms.

The Fourier's coefficients are to be evaluated using the above formulas. We have

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\left[\int_0^{T/2} a\,dt - \int_{T/2}^{T} a\,dt\right]$$

$$= \frac{2}{T}\left[a\frac{T}{2} - a\frac{T}{2}\right] = 0$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt$$

$$= \frac{2}{T}\left[\int_0^{T/2} a\cos n\omega t\,dt - \int_{T/2}^{T} a\cos n\omega t\,dt\right]$$

$$= \frac{2}{T}\frac{a}{n\omega}\left[\left[\sin n\omega t\right]_0^{T/2} - \left[\sin n\omega t\right]_{T/2}^{T}\right]$$

$$= \frac{a}{n\pi}\left[\left[\sin n\pi - \sin 0\right] - \left[\sin 2n\pi - \sin n\pi\right]\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt$$

$$= \frac{2}{T}\left[\int_0^{\frac{T}{2}} a\sin n\omega t\,dt - \int_{T/2}^{T} a\sin n\omega t\,dt\right]$$

$$= \frac{2}{T}\frac{a}{n\omega}\left[-\left[\cos n\omega t\right]_0^{T/2} + \left[\cos n\omega t\right]_{T/2}^{T}\right]$$

$$= \frac{a}{n\pi}\left[-\left[\cos n\pi - \cos 0\right] + \left[\cos 2n\pi - \cos n\pi\right]\right]$$

If n is even number, $\cos n\pi = 1 = \cos 2n\pi$.

$\therefore b_n = 0$

If n is odd number, $\cos n\pi = -1$, $\cos 2n\pi = 1$.

$\therefore b_n = \frac{4a}{n\pi}$.

Fourier's series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

In this case $a_0 = 0$, $a_n = 0$, for all values of n. $b_n = 0$ for all even values of n. $b_n = \frac{4a}{n\pi}$ for all odd values of n.

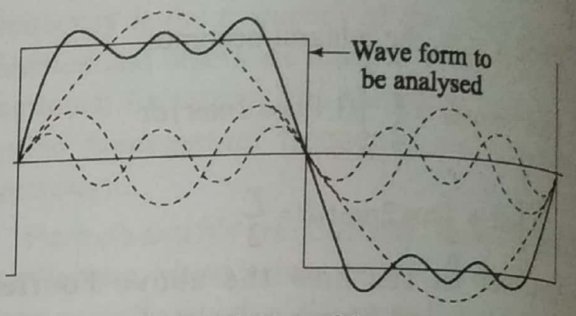
Substituting these values of the constants in the Fourier's series we get

$$f(t) = \frac{4a}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + ...\right) \qquad ....3.29$$

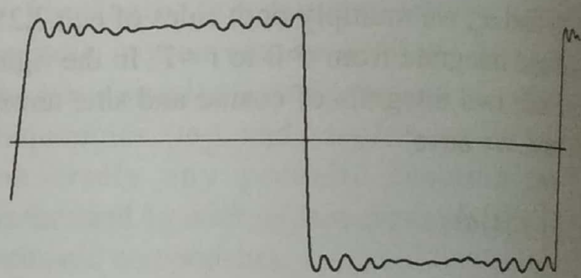
We have been able to express the given periodic function as the sum of a series of simple harmonic functions having *different* amplitudes and angular frequencies ω, 3ω, 5ω,.....We see that only odd harmonics are present.

The *more and more terms we add, the nearer and nearer would be the sum to the given function.* The amplitude of a given term represents the weight or contribution of that term.

In Fig. 3.9(a) we see the curve corresponding to the first three terms of the above series separately and also their sum. In Fig. 3.9(b) we see the curve obtained by summing first fifteen terms. We should be convinced that the above statement is true.



(a) Addition of three terms



(b) Addition of fifteen terms

Fig. 3.9

We also notice that addition of more and more harmonics changes the *shape* of the resultant function and ultimately we get back the original curve.

(2) Saw tooth Wave

In Fig.3.10a,we see a saw tooth wave. The function increases linearly from 0 to a in time T,

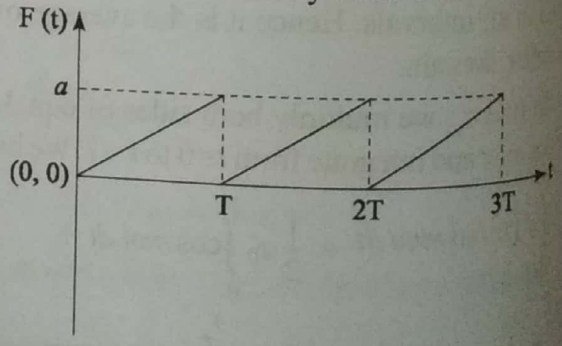


Fig. 3.10(a)

then drops sharply to zero. This is repeated. Hence it is a periodic function, time period is T. But the function has *finite discontinuity* after each interval of time T.

At any instant t within time interval 0 to T, the value of the function $f(t)$ has value given by

$\frac{f(t)}{t}=\frac{a}{T}$. $\therefore f(t)=\frac{a}{T}t$.

Now let us find the Fourier's coefficients of the Fourier's series.

$$a_0=\frac{2}{T}\int_0^T f(t)dt=\frac{2}{T}\int_0^T\frac{a}{T}t\,dt=\frac{2}{T}\frac{a}{T}\int_0^T t\,dt=\frac{2a}{T^2}\cdot\frac{T^2}{2}=a$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt=\frac{2}{T}\frac{a}{T}\int_0^T t\cos n\omega t\,dt\cdot$$

Integrating by parts, we get

$$a_n=\frac{2}{T}\frac{a}{T}\int_0^T t\cos n\omega t\,dt=\frac{2a}{T^2}$$

$$\left[\left[\frac{t\sin n\omega t}{n\omega}\right]_0^T-\int_0^T\frac{\sin n\omega t}{n\omega}dt\right]$$

$$=\frac{2a}{T^2}\left[0+\left[\frac{\cos n\omega t}{n^2\omega^2}\right]_0^T\right]$$

$$=\frac{2a}{T^2}\left[\frac{\cos 2n\pi-\cos 0}{n^2\omega^2}\right]=\frac{2a}{T^2}[0]=0$$

$$b_n=\frac{2}{T}\frac{a}{T}\int_0^T t\sin n\omega t\,dt=\frac{2a}{T^2}$$

$$\left[\left[-\frac{t\cos n\omega t}{n\omega}\right]_0^T+\int_0^T\frac{\cos n\omega t}{n\omega}\right]$$

$$=\frac{2a}{T^2}\left[-\frac{T\cos 2n\pi}{n\omega}+\left[\frac{\sin n\omega t}{n^2\omega^2}\right]_0^T\right]$$

$$=\frac{2a}{T^2}\left[-\frac{T}{n\omega}+0\right]=-\frac{a}{\pi n}$$

Fourier's series is

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega t+b_n\sin n\omega t\right)$$

In this case $a_0=a$, $a_n=0$, , for all values of

$n.b_n=-\frac{a}{\pi n}$

Substituting these values of the constants in the Fourier's series we get

$$f(t)=\frac{a}{2}-\frac{a}{\pi}\left(\sin\omega t+\frac{1}{2}\sin 2\omega t+\frac{1}{3}\sin 3\omega t+....\right)$$

............3.30

In Fig. 3.10(b) we see the curve obtained by summing first three terms. In Fig. 3.10(c) we see the curve obtained by summing first fifteen terms.

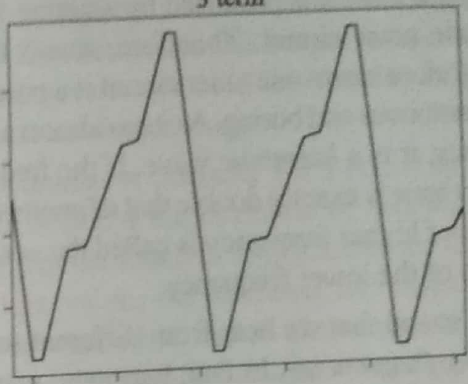


Fig. 3.10(b)

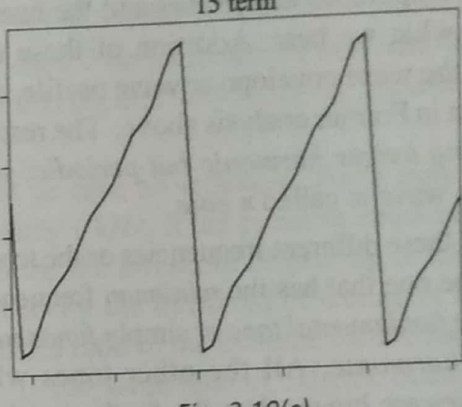


Fig. 3.10(c)

Again we find that addition of more and more harmonics changes the *shape* of the resultant function and ultimately we get back the original curve.

3.6 Characteristics of musical sound

Sound that we hear may be divided broadly into two categories.

(1) *Noise* : It is the sound which produces unpleasant sensations. It is generally the sound of short duration and is sharp and abrupt. It is produced by abrupt vibrations like a bomb explosion, click of a hanger on an anvil, etc. Sometimes noise is taken as the totality of undesired sound. Noise is *disordered* sound; it contains sound of *all frequencies within acertain range* having *no noticeable dominant frequency.*

(2) *Musical sound or note* : It is the sound which generally produces pleasing sensation. It is *continuous* and has a regular *periodicity*. Music is *ordered* sound. It consists of sound of discrete and rational frequencies; the ratios are simple fractions. And there is a *noticeable dominant frequency.*

We first study the structure of the different musical sound that we hear.

Sound of a *single* frequency is called a *pure tone* or simply a *tone*. It is produced by a tuning fork or electronic tuner circuit. Therefore, sound from a tuning fork or electronic tuner circuit is a pure tone; it is monotonous and boring. As it has almost a single frequency, it is a *harmonic* wave. If the frequency of a pure tone is exactly double that of another, then the tone of higher frequency is called the *octave* of the tone of the lower frequency.

The sound that we hear from different sources has *many frequencies*. In fact, harmonic waves of many different frequencies superpose with different amplitudes and phases and the resultant wave is what we hear. Addition of these tones changes the wave envelope or wave profile, as we have seen in Fourier analysis above. The resultant wave is *no longer harmonic but periodic*; sound of such a wave is called a *note*.

Of all these different frequencies or the tones in a note, the one that has the *minimum* frequency is called the *fundamental tone* or simply *fundamental or first harmonic*. All the other tones whose frequencies are bigger than the fundamental tone are called the *overtones*. Those overtones whose frequencies are *integral* multiple of the fundamental tone are called *harmonics*. So when we hear a note, we hear the fundamental and the harmonics and other overtones all blended together.

Now we are in a position to study the characteristics of musical sound.

Musical sound has *three* essential characteristics : (1) *Loudness,* (2) *Pitch,* (3) *Quality* or *timbre*.

If two musical sounds has all the above three characteristics same, then there is nothing to distinguish one from the other. All these three characteristics are judged by *sensations* they produce, and to some extent *interdependent*. Also it varies to some extent from person to person. But still each of these is largely governed by one or more distinct *physical factors*. We are going to study the *physical causes* on which these characteristics depend.

(1) *Loudness* : It is a *sensation* that depends on the amount of *energy* falling per *unit area* in *unit time* in our ear; greater the energy louder is the sound. Obviously loudness is related with the intensity of the wave. Intensity of a wave, we know, is the flow of energy normally through unit area per unit time.

Loudness *increases* with intensity, but is *not proportional* to the intensity. Moreover the same intensity at widely different frequencies may produce sensations of different loudness. Here we like emphasize that loudness of sound is an *aural* sensation and a physiological phenomenon. Intensity, on the other hand, is a physical quantity and is the main external cause for the sensation of loudness.

Naturally loudness depends on the physical factors on which intensity depends. In chapter 2 we have calculated the intensity (I) of a harmonic wave. We got

$$I = 2\pi^2 \rho c n^2 a^2$$

We see that intensity (I) of a harmonic wave depends on the density (ρ) of the medium, speed (c), frequency (n) and amplitude (a) of the sound.

Other factors on which intensity depends are

(i) The *size of the source*, greater the size, greater is the loudness. The reason is that a body of greater size can produce vibration of greater mass of the medium surrounding it, producing bigger disturbances.

(ii) The *distance* (r) *of the source* from the observer. The relation can easily be found as follows: Let S be a small source emitting energy Q per second. When it travels a distance r from S, the energy is spread over the surface of a sphere of radius r with S at its centre,

$\therefore$ Energy passing per unit area per unit time radially (i.e., normally) at a distance r from the source is the intensity given by

$$I = \frac{Q}{4\pi r^2} \quad \therefore \; I \propto \frac{1}{r^2} \qquad \text{3.31}$$

(iii) *Presence of other bodies* near to it : We have seen that when a tuning fork is pressed on a table, the intensity increases or the hollow wooden box in string instruments enhances the intensity, both because of forced vibration. If there is body which resonates with the frequency of the sound there is a marked increase in intensity.

(iv) *Presence of a reflector* at the right position : The reason is obvious; some of extra energy arrives there by reflection.

(2) *Pitch* : It is the characteristic by which we distinguish a *shrill* or sharp sound from a *dull* or flat sound. It is that characteristic which changes as we press the keys of a harmonium one after another; the pitch of the sound becomes more and more shrill as we go from the first key 'Sa' to the second key 'Re', to the third key 'Ga', and so on.

Pitch of a note is decided by the *fundamental* tone. Higher the fundamental tone, shriller is the note. Pitch is *almost proportional* to the fundamental frequency.

The situation is, however, not so simple. It is found that the pitch of a sound also depends on the *overtone structure* of the note. The loudness of sound also affects the pitch up to 1000 Hz. An increase in loudness causes a decrease in pitch. From about 1000 to 3000 Hz, the pitch is independent of loudness, while above 3000 Hz an increase in loudness causes an increase in pitch.

Pitch of a sound also depends on the *relative velocity between the source and the observer*. This fact is called the Doppler's effect.

(3) *Quality or timbre* : It is that characteristic by which we can distinguish notes of the *same pitch and loudness* coming from different sources, say one from a flute and the other from a violin. The quality of a musical sound is due to the presence of overtones in addition to the fundamental. The *number, nature and relative intensities* of the overtones determine the quality of the musical sound. *Greater* is the number of harmonics present in a note, musically *richer* it is. Addition of more and more harmonics changes the quality (wave profile) of the sound but does not change the pitch; pitch is decided by the fundamental tone.

Our ears and brains are capable of recognising the subtle differences produced in the musical quality of sound due to slight variations in number and relative intensities of the different overtones and harmonics.

3.7 Intensity thresholds, intensity levels and units

We know that normal human ear can hear sound only if its frequency is within the limit of 20 Hz to 20,000 Hz. But within this range, to hear a sound its *intensity* must lie between certain ranges. We shall now study this range.

The *minimum* intensity for a sound to be *just audible* is called the *threshold of hearing*. If the intensity of the sound is slowly increased from this minimum value, the loudness increases until a value is reached at which the character of the sensation changes and becomes one of *pain*. This value of intensity is called the *threshold of feeling* or *threshold of pain*.

It should be remembered that both these two thresholds depend very much on *frequency* of the sound.

Let us take an example. For a pure tone of frequency 1 kHz, if the pressure *amplitude* of the sound for the threshold of hearing is taken as 1, then that for the threshold of feeling is 10^7. That means the ratio of the extreme *intensities* is about $10^{14} : 1$. The same ratio of extreme intensities for sound of frequency 80 Hz is about $10^8 : 1$.

It also varies from person to person and on some other factors. For example, younger people are found to be more tolerant of loud sound than elder people.

What we should notice is the *enormous range* of intensity in which our ear can respond and that the two limits *depend on frequency* of sound. In Fig. 3.11, we see the above lower and upper

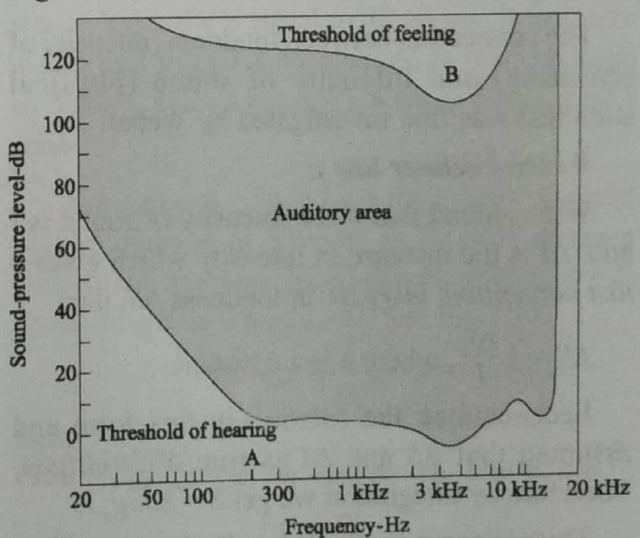


Fig. 3.11

thresholds A and B respectively for different frequencies. Also we see from the figure that our ear is most sensitive around 3 kHz.

Pressure amplitude (p_0) of sound wave can be *measured*. From this data we can calculate the displacement amplitude (a)and intensity (I) by the relations $p_0 = 2\pi a n c\rho$ [see eqn.2.7] and $I = 2\pi^2 a^2 n^2 \rho c$ [see eqn.2.15]. We can take $\rho = 1.29$ kgm^{-3} and $c = 345$ ms^{-1}.

Now we quote some data for sound waves of frequency 400Hz.

The *faintest* sound of this frequency that can be heard has pressure amplitude, $(p_0)_1 = 8 \times 10^{-5}$ Nm^{-2}, displacement amplitude $a_1 = 7.15 \times 10^{-11}$ m and intensity $I_1 = 7.19 \times 10^{-12}$ Wm^{-2}.Notice a_1 is of the order of *molecular* dimensions and is much smaller than the average molecular separation in a gas. From these data we can realise the *remarkable sensitivity* of our ears.

The *loudest* sound of this frequency that produces sensation of pain or discomfort in our ears has $(p_0)_2 = 30$ Nm^{-2}, $a_2 = 1 - 2$ mm and $I_2 = 1$ Wm^{-2}.We know normal atmospheric pressure is 1.013×10^5 Nm^{-2}.

Now we can see the *enormous range* of displacement and pressure amplitudes and intensity over which our ears can respond. No man-made apparatus can cover such a wide range.

3.7.1 *Intensity and loudness* :

We have seen above that *perception* of loudness is *not* proportional to intensity of the sound. Loudness is found to vary as the *logarithm* of the intensity.

The connection between loudness (intensity of sensation) and Intensity of sound (physical stimulus) was first investigated by Weber.

Weber-Fechner law :

Weber found that if the intensity of sound is I and ΔI is the increase in intensity which gives a *just perceptible increase* in loudness ΔS, then

$$\Delta S = k\frac{\Delta I}{I}, \text{ where } k \text{ is a constant.}$$

Fechner took the relation in this form and assumed that ΔS and ΔI as true differentials. Therefore by integration we get $S = k \log_e I$.

This is known as the Weber-Fechner law. This law holds approximately not only for the sensation of loudness, for all sensations, like sense of weight, intensity of light, etc,.

3.7.2 *Intensity level, bel and decibel* :

From Weber-Fechner law we define *intensity level* of a sound and its *unit*.

For this purpose we take the constant $k = 1$ and the logarithms are taken to base ten.

In perception of sound, *relative* intensity is more important than *absolute* values of intensities. Hence intensity (I) is generally measured as the *ratio* to a *standard minimum* intensity I_0.

If loudness for intensities I and I_0 be S and S_0 respectively, then we have

$$S = \log_{10} I \text{ and } S_0 = \log_{10} I_0$$

Intensity level (L) of the sound of intensity I is, by definition,

$$L = S - S_0 = \log_{10}\left(\frac{I}{I_0}\right).$$

Hence intensity level, $L = (S - S_0)$ tells how much the loudness of a given sound is *above* the standard *minimum* value (reference intensity).

The *standard minimum* intensity has been *chosen* to be $I_0 = 10^{-12}$ watts / m^2. It is about the intensity that can *just* be heard at frequency 1 kHz. Our ears are very sensitive at this frequency. The name given to this unit of intensity level is bel. Notice, it is dimensionless.

Therefore intensity level L of sound of intensity I is

$$L = \log_{10}\left(\frac{I}{I_0}\right) \text{bel.} \quad ....3.32$$

Suppose intensity of a sound is $I = 10I_0$, then its intensity level is

$$\therefore L = \log_{10}\left(\frac{I}{I_0}\right) = \log_{10}\left(\frac{10I_0}{I_0}\right) = \log_{10} 10 = 1$$

$\therefore$ Intensity level of a sound is 1bel means it is *ten times* the threshold intensity I_0.

The unit in common use is Decibel (dB). 1 dB = 0.1 bel.

$\therefore$ Intensity level of a sound wave measured in *decibel* is given by

$$\beta = 10\log_{10}\left(\frac{I}{I_0}\right) \text{dB} \quad ....3.33$$

If intensity level of a sound is 1 dB we have from eqn. 3.33

$$\beta = 1 = 10\log_{10}\left(\frac{I}{I_0}\right)$$

or, $\log 10\left(\frac{I}{I_0}\right) = 0.1$. $\therefore \left(\frac{I}{I_0}\right) = 1.2589 \approx 1.26$

$\therefore$ Intensity level of a sound is 1 dB means it is 1.26 times the threshold intensity I_0.

In other words, intensity level alters by 1dB when intensity of sound changes by 26%

In this scale, intensity level of the *threshold of hearing* is $\beta = 10\log_{10}\left(\frac{I_0}{I_0}\right) = 0$

At the threshold of pain we take, $I \approx 10^{12} I_0$. Therefore intensity level of the *threshold of pain* is $\beta = 10\log_{10}(10^{12}) = 120$ dB

Thus 0 dB to 120 dB is the *whole range of sound intensity level* that we can hear. In the Table-I below we can see the intensity levels of some common sounds.

Table I

Source	dB
Normal breathing	10
Rustling leaves	20
Soft whisper	30
Normal conversation	60-70
Heavy truck	90
Loud siren	100
Thunder	110
Rock concert with amplifiers	120
our eardrums brust	160

We know that for harmonic waves intensity is proportional to square of amplitude. For sound waves in air, the wave amplitude is the maximum pressure, measured relative to undisturbed atmospheric pressure. Thus in terms of pressure amplitudes we can write $\frac{I}{I_0} = \frac{p^2}{p_0^2}$.

We can have *alternate definition* of intensity level for sound waves :

$$\beta = 20\log_{10}\left(\frac{p}{p_0}\right) \text{dB}$$

Value of p_0 is $p_0 = 2 \times 10^{-5}$ N/m², the threshold for hearing at 1000 Hz.

3.7.3 *Phon :*

Intensities and intensity level in dB are *measurable* (objective) quantities. Loudness, on the other hand, is *subjective*. When we express intensity level of a sound in dB unit, we ignore the fact that the human hearing sensitivity *varies with frequency*.

For example, a 60 dB sound with a frequency of 1000 Hz sounds louder than a 60 dB sound with a frequency of 500 Hz. Hence a new unit, called phon, was introduced to indicate individual's perception of loudness.

Definition : Loudness of any sound in phons is *numerically equal* to the intensity level in *decibels* of an *equally loud* pure tone of frequency 1 kHz (standard source).

Therefore if loudness of a sound is 1 phon, then it is as loud as the sound of intensity level 1 dB at 1 kHz, a sound of loudness of 60 phon is as loud as 60 dB at 1 kHz, etc.

3.7.4 *Sone :*

It should be realised that when phon is used as a unit of loudness, we do not get a *linear* scale, which is *proportional* to loudness.

For example, 40 phons sound is *not twice* as loud as 20 phons sound. It was, however, observed that a 10 *phon increase* in sound level is most often perceived as a *doubling of loudness*. From this observation the sone scale of loudness was created.

Definition : 1 sone sound is defined as a sound whose loudness is equal to 40 phons. It is a very useful unit in that 2 sone sound is twice as loud as 1 sone, 4 sone sound is 4 times as loud as 1 sone.

Explanation : 50 = (40+10×1) phons would have a loudness of 2 sones, as 10 phon increase doubles the loudness. Similarly, 60 (=40 + 10 ×2) phons would be 4 sones, etc. See Fig 3.12.

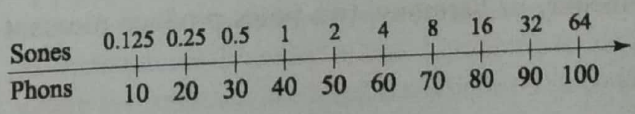


Fig. 3.12

To get the relation between sone and phon, we observe that when loudness is 50 phon, we first subtract 40 from 50, and then divide the difference by 10. We get 1, then the loudness in sone is $2^1 = 2$.

Therefore if loudness of a sound has values P phons and S sones in the two units, then relation between these two can be written as $S = 2^{\frac{P-40}{10}}$

3.7.5 *Musical Scale :*

In music *absolute values* of the *frequencies* of the notes are *not important*. In passing from one note to another, our ears are *sensitive to* the *ratio* in which their frequencies change, not to their numerical difference. A doubling from 100 Hz to 200 Hz sounds similar to a doubling from 800 Hz to 1,600 Hz.

Musical interval : The *ratio of frequencies* of any two notes is known as the *musical interval* between them.

Notes of the same frequency are said to be in *unison*. If the ratio of two notes is 2/1, the interval between the two notes is called *octave*, whatever be their actual frequencies. Therefore interval between 100 Hz and 200 Hz or between 1000 Hz and 2000 Hz is an octave.

Similarly, interval is called *fifth*, *fourth*, *major third* and *minor third* if the ratio is 3 / 2, 4 / 3 , 5 / 4 and 6 / 5 respectively.

Consonance and Dissonance :

When two or more notes are sounded together, their combination is called a *chord*. If the combination produces musically pleasing sensation to the ears, the two are called *concord*. The pleasant effect is called *consonance*. If the combination produces musically disagreeable or jarring effect upon the ears, it is called a *discord* and the disagreeable effect is called *dissonance*.

When two notes which produce concord are sounded together, the pleasing effect is called *harmony*. When they are sounded one after the other, the effect is called *melody*. Whether it is melody or harmony, two notes produce pleasant effect if the ratio of their frequencies can be expressed as fraction of small integers.The smaller is the integer the more pleasing is the effect.

In other words, the musical intervals which are perceived to be most consonant are composed of *small integer ratios* of frequency. Such ratios are known as "just". Four intervals are particularly important: the octave (2/1), a just fifth (3/2), a just fourth (4/3), and a just major third (5/4).

The reason behind the dissonance is formation of *beats*. When two notes are produced together, *beats* are formed between the fundamental frequencies or between the overtones. This breaks up the resultant sound into pulses, which produces the irritation in our ears. Flickering of light produces similar unpleasant sensations in our eyes.The range over which the dissonance due to the beats persists is different for different frequencies.

Musical scale : A musical scale, a *set of* notes with consonant intervals,is used to make music. Musical scales are defined in terms of the *frequency ratio* of each note to a *reference pitch*, called the root or keynote of the scale.

Diatonic scale : It is a musical scale formed by introducing *six notes* between a given note and its octave. Thus the octave is divided into *seven* intervals. The notes are so chosen that these are consonant among themselves and with the extreme notes of the octave.

The eight notes are indicated by the letters: C, D, E, F, G, A, B, c. Frequency of c is twice that of C, hence note c is an octave higher than keynote C.In music, several such octaves extending on both sides of the keynotes are required. For example, for the next higher octave c would be first member and the last member c′ is an octave higher than c and two octaves higher than C.

In Table -II below, we can see the names of the successive notes, their relative frequencies and intervals between successive notes. Here keynote is chosen as 256 Hz, but it can have any chosen value.

Table II

Symbol	C	D	E	F	G	A	B	c
European Indian	DO SA	RE RE	MI GA	FA MA	SOL PA	LA DHA	TI NI	do sa
Relative frequency	1	$9/8$	$5/4$	$4/3$	$3/2$	$5/3$	$15/8$	2
Interval		$9/8$	$10/9$	$16/15$	$9/8$	$10/9$	$9/8$	$16/15$
Frequency on the basis that C = 256	256	288	320	341	384	427	480	512
Relative frequency	24	27	30	32	36	40	45	48

Tempered scale :

Diatonic scale represents absolute harmonic perfection. The ratios are so chosen that no beats are produced. But it has one very important disadvantage.

As a result of the ratios used in this scale, this tuning system can only apply to one keynote at a time. For example, suppose instead of using the C of the diatonic scale as the keynote, we choose D or E as the keynote. Then new notes not given by diatonic scale will arise if the diatonic intervals are used. This is shown in Fig.3.13. In Table-III we can see comparison of the two scales.

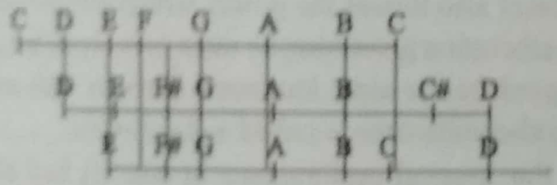


Fig. 3.13

Though many are not too apart, there are discrepancies. The discrepancies become worse as more and more transpositions are applied. Hence in diatomic scale music can only be performed in one scale; all exciting musical elements such as transpositions, modulations cannot be performed. This fact led to the development of a new system of musical intervals. This is called 'tempered scales'.

In diatonic scale there are eight notes in an octave and the intervals are not equal. In tempered scale there are twelve notes in an octave and the interval between successive notes are *all equal*. Therefore if X be the interval between the successive notes of this scale, then

$$X^{12} = 2. \therefore X = 2^{1/12} = 1.059463$$

The interval between any two consecutive notes is 1.059463. In Table III, we can compare the two scales; the values are all rounded to third decimal. We see that there are five new notes in the new scale, which are indicated by stars in the table. We notice that the intervals on the two scales do not differ by even 1%. Hence the concords of a diatonic scale may be reproduced on the tempered scale.

Table III

	DO		RE		MI	FA		SOL		LA		TI	do
Diatonic	1		1.125		1.250	1.333		1.500		1.667		1.875	2
Tempered	1	*	1.122	*	1.260	1.335	*	1.498	*	1.682	*	1.883	2

The main advantage of the new scale is that any of the notes can be taken as the keynote and still give all the intervals necessary for concord. In instruments like the piano, harmonium etc., where the intervals are fixed the tempered scale is indispensable.

3.8 Acoustics of buildings

Acoustic of building or architectural acoustics deals with the design and construction of buildings and rooms to produce an ideal situation of hearing experience.In buildings mainly noises, both internal and external, should be reduced to minimum. In closed rooms, in particular in different auditoriums, there are different kinds of disturbances, which are to be reduced to minimum. Hence behaviours of sound wave in closed spaces are also studied in this discipline.

Here we are concerned with some of the disturbances we experience when there is musical performance or lecture in a closed room and how to minimise them.

Most important acoustical defect of an auditorium is reverberation. We shall consider it a little detail.

3.8.1 *Reverberation :*

When a sound is produced in a closed room, sound waves are reflected back from the walls, floor and ceiling. If absorption of sound by the different surfaces is small,the waves are reflected back and forth from one surface to another. As a result of such multiple reflections, the effect of the sound may *persist for a long time* after it is produced. The fresh sound produced during this time is blended with the former and the two cannot be distinguished; clarity is lost. This defect is particularly important in a lecture theatre.

This *prolongationor persistence* of audible sound even after the source has ceased to produce sound is called *reverberation.*

Naturally this effect can be reduced to a great extent by using suitable absorbing materials in different reflecting surfaces.

The level of reverberation in a hall is measured by reverberation time. It was first introduced by W.C. Sabine.

3.8.2 *Reverberation time :*

It is defined as the time (T) required for a sound to diminish from its initial intensity (I_1) to *one-millionth* of that intensity.

If after time T the intensity I_1 reduces to I_2, then, from the definition, we have $\frac{I_2}{I_1}=10^{-6}$

Initial intensity level in decibel was

$$\beta_1 = 10\log_{10}\left(\frac{I_1}{I_0}\right)$$

After time T, intensity drops to

$$\beta_2 = 10\log_{10}\left(\frac{I_2}{I_0}\right)$$

∴ Drop in intensity level in decibel,

$$\beta_1 - \beta_2 = 10\left(\log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right)\right)$$

$$= 10\log_{10}\left(\frac{I_1}{I_2}\right) = 10\log_{10}10^6 = 60$$

Hence in reverberation time intensity level [drops] by 60 dB. Change in decibel level is -60 dB.

Therefore if the initial level was 60 dB, the sound level after the reverberation time [will] cease to be audible.

Reverberation has both disadvantage [and] advantage :

If reverberation time is large, sound pe[rsists] for a long time. As a result, there is overlappi[ng of] sounds of two syllables uttered in succe[ssion] which produces confusion. The acoustic cond[ition] will be bad.In a *lecture theatre*, in particular, [a small] reverberation time is preferred.

On the other hand, if the reverberation ti[me is] too small, any sound produced will end abr[uptly] and the sound becomes unpleasant to hear. [The] speaker also misses the power which a little m[ore] reverberation gives him; he must now work ha[rder] to produce the same loudness. A room with [no] reverberation time is called a *dead room.*

For musical performance in concert hall [etc.] the right reverberation time is to be chosen. [For] example, if the music depends for its effe[ct on] precision of detail, a short time of reverberati[on is] to be preferred. For music whose effect depe[nds] on massiveness and power, a longer tim[e of] reverberation is required. In fact right amou[nt of] reverberation can drastically enhance the s[ound] quality, when a musical symphony or orches[tra is] being played in a hall.

Optimum reverberation time is 1s *for sp[eech]* and 2s *for music*.Hence the required reverber[ation] time for a room depends on the use the ro[om is] intended for.

Absorption coefficient :

Absorption coefficient of a surface is the of the amount of sound energy *absorbed* by the surface in a given interval of time to the amount of sound energy *incident* on it in that time interval. It is independent of the intensity of the sound.

Open window is a perfect absorber. Its absorption coefficient is 1. Absorption coefficient of 1 square foot area of an open window is defined as 1 *sabin*. If the metric unit is used then absorption coefficient of 1 m^2 is called the metric sabin.

It is interesting to note that a listener is equivalent to an open window of area of about 5 square foot.

3.8.3 *Sabine formula*:

Sabine developed the theory of reverberation and derived an expression for the reverberation time in terms of size of the hall, area and absorption coefficient of the various surfaces in the hall. This relation is known as Sabine formula, which is given by

Reverberation time, $T = \dfrac{0.617V}{aS}$ 3.34

Here V is the volume of the hall, a = mean absorption coefficient of the different surfaces in the hall and S is the total surface area of all the surfaces in the hall. Here 0.167 has the unit s/m. V is in m^3 and S in m^2, so T is in s; see below.

Now we shall study a simple derivation of the above formula.

Let S_1, S_2, S_3...be areas of different surfaces present in a hall having absorption coefficients a_1, a_2, a_3,.....Then *mean* absorption coefficient is given by

$$a = \frac{a_1S_1 + a_2S_2 + a_3S_3 + ...}{S_1 + S_2 + S_3 + ...} = \frac{\sum a_iS_i}{S}$$

Let E be the energy per unit volume present at an instant in the hall and dE is the fall in the energy due to absorption in time interval dt.

$dE = -anEdt.$ ....(i)

Here n is the number of reflections of sound wave in unit time.

If V is the volume of the hall and S is the total area of reflecting surface, then it can be shown that sound travels a mean distance of $4V/S$ between two successive reflections.

If v is the velocity of sound, then time interval between two successive reflections is given by,

$$\tau = \frac{4V}{S} \times \frac{1}{v}$$

∴ Average number of reflections per second is

$$n = \frac{1}{\tau} = \frac{Sv}{4V}$$

Substituting the value of n in eqn.(i), we get

$$dE = -a\frac{Sv}{4V}Edt$$

or $\dfrac{dE}{E} = -a\dfrac{Sv}{4V}dt$ ....(ii)

Let the initial energy density of sound when the source is cut off be E_0 and that after time t is E. That means at $t = 0$, $E = E_0$ and at $t = t$, $E = E$. To get the relation between E and E_0, we integrate relation (ii) within these limits.

$$\int_{E_0}^{E} \frac{dE}{E} = -a\frac{Sv}{4V}\int_0^t dt$$

or, $\log_e\left(\dfrac{E}{E_0}\right) = -a\dfrac{Sv}{4V}t$

∴ $\dfrac{E}{E_0} = e^{-a\frac{Sv}{4V}t}$ ....(iii)

If T is the reverberation time, then from definition we have: when $t = T$, $E/E_0 = 10^{-6}$.

Therefore from eqn. (iii) we have

$$e^{-a\frac{Sv}{4V}T} = 10^{-6}$$

or, $a\dfrac{Sv}{4V}T = 6\log_e 10 = 6 \times 2.303$

or, $T = 6 \times 2.303\dfrac{4V}{Sv}$

Taking velocity of sound $v = 330$ m/s, we get

Time of reverberation, $T = \dfrac{6 \times 2.303 \times 4}{330}\dfrac{V}{aS}$

$$= \frac{0.167V}{aS} = \frac{0.167V}{\sum a_iS_i}$$

Notice, 0.167 has the unit s/m. V is in m^3 and S in m^2. Therefore T is found in s.

This is Sabine's formula.

3.8.4 *Requirements for good acoustics in a hall and auditorium*:

(1) Each syllable of the sound produced in the hall should be sufficiently loud and intelligible at every part of the hall. Uniform distribution of loudness in a hall is essential for satisfactory hearing. This can be done by using loud speakers. In absence of loud speakers, large sounding boards may be kept behind the which are multiples of the frequency of given function.

Physical implication is that superposition of simple harmonic motions of correct frequencies and amplitudes can produce any periodic motion, however complicated it may be. Simple harmonic motions are the building blocks of periodic motion.

Tone : Sound of a single frequency is called a tone.

Musical sound or note: It is the sound produced by continuous vibration having a regular periodicity. Generally such a sound produces pleasing sensation. All musical sounds contains many tones.

Three characteristics of musical sound: loudness, pitch and quality or timbre.

Though these three are essentially sensations and to some extent interdependent, they individually depend on several physical factors.

Loudness depends on intensity and sometimes on frequency of the sound wave. Loudness is not proportional to intensity.

Pitch is the characteristic by which we distinguish a shrill sound from a dull one. It is determined by the fundamental tone (minimum frequency) present in a sound. Pitch is *almost proportional* to the fundamental frequency, but overtone structure and intensity affect pitch.

Quality is the characteristic by which we can distinguish sounds of the same loudness and pitch coming from different sources. Quality is mainly decided by the overtones and harmonics present in a sound and their relative intensities.

Range of frequency we can hear: For normal human ear: From 20 Hz to 20,000 Hz.

Threshold of audibility: It is the minimum intensity for a sound to be just audible.

Threshold of feeling: It is the minimum intensity above which the character of the sensation changes and becomes one of pain or discomfort in our ears.

Both these two limits depend on the pitch of the sound.

Ratio of these two extreme intensities is very large: at frequency 1 kHz the ratio is $10^{14} : 1$ and at frequency 80 kHz it is $10^{8} : 1$. The range is indeed enormous.

Weber-Fechner law: Loudness (S) varies *logarithm* of the intensity (I): $S = k \log I$

Intensity level of a sound : If Intensity intensity level is

$$L = S - S_0 = k \log_e \left(\frac{I}{I_0}\right)$$

I_0 is the intensity of a standard intensity.

Bel and decibel : In this unit standard in is chosen to be $I_0 = 10^{-12}$ watts / m². We take and logarithms are taken to base ten. Intensity in bel unit is defined as

$$L = \log_{10}\left(\frac{I}{I_0}\right) \text{ bel.}$$

The unit in common use is Decibel (dB) = 0.1bel.

In decibel unit intensity level of a sou intensity I is

$$\beta = 10 \log_{10}\left(\frac{I}{I_0}\right) \text{ dB}$$

Intensity level of the *threshold of hear*

$$\beta = 10 \log_{10}\left(\frac{I_0}{I_0}\right) = 0.$$

Intensity level of the *threshold of pain*

$$10 \log_{10}\left(\frac{1}{10^{-12}}\right) = 120 \text{ dB.}$$

Thus 0 dB to 120 dB is the *whole ran sound intensity level* that we can hear.

Unit of loudness : Intensities and in levels are measurable quantities and independent of human perceptions of sound.

Loudness, on the other hand, is subje experience of the sound. In fact two sounds same decibel level but of different freque appear to have different loudness to huma Therefore loudness should have a differen which may take into account the depende loudness on frequency. Phon is the desired loudness. To define it we take a standard of sound of frequency 1 kHz.

Loudness of a sound is x phon means has the same loudness as that of the s frequency 1 kz having intensity level of x

But the new unit phon is not fully satis because in this we do not get a linear scale of loudness 40 phon is not twice as loud as the sound of loudness 20 phons. Hence another unit of loudness of sound was created. This unit is called sone.

1 sone sound is defined as a sound whose loudness is equal to 40 phons.

Musical interval between two notes: It is the ratio of frequencies of the two notes.

Octave: The interval between the two notes is called octave, if the ratio of two notes is 2 / 1.

Musical scale: It is a set of notes with consonant intervals, used to make music.

The two most important musical scales are diatonic scale and tempered scale.

Reverberation: Prolongation or persistence of audible sound even after the source has ceased to produce sound is called reverberation.

The time required for a sound to diminish from its initial intensity to one-millionth of that intensity is defined as the reverberation time.

Reverberation has both disadvantage and advantage.

Sabine formula : $T = \frac{0.167V}{\sum a_i S_i}$.

Requirements for good acoustics in a hall and auditorium: discussed in detail in the text.

SOLVED PROBLEMS

1. *A pendulum 20 m long requires 1 hour for its amplitude to be reduced half its original value. Find the damping factor and time period.*

ANS As amplitude is reduced to half its original value (a), we have

$\frac{a}{2} = ae^{-b\times 3600}$.

$\therefore b = \frac{\log_e 2}{3600} = 19.3 \times 10^{-5}$ per second.

Natural angular frequency (ω) is that of a simple pendulum.

$\therefore \omega = \frac{T}{2\pi} = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.8}{20}} = 0.7$ radian per second.

$\therefore$ Damped angular frequency

$\omega_d = \sqrt{\omega^2 - b^2} \simeq \omega$, as $b^2 << \omega^2$.

2. *A very small body of mass 0.130 kg is undergoing a damped simple harmonic motion. The force constant is 1.20 N/m and retarding force per unit velocity is 0.11 kg/s. Write down the equation of motion. Find whether the motion is overdamped or underdamped. Find what should be the retarding force per unit velocity in order that the motion is critically damped.*

ANS Restoring force is $F_r = -kx = -1.2x$, damping force, $F_d = -0.11\frac{dx}{dt}$. Therefore the equation of motion is

$$0.13\frac{d^2x}{dt^2} = -0.11\frac{dx}{dt} - 1.2x$$

or $$\frac{d^2x}{dt^2} + \frac{0.11}{0.13}\frac{dx}{dt} + \frac{1.2}{0.13}x = 0$$

Comparing with the equation :

$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0$, we get $b = \frac{0.11}{0.13}\times\frac{1}{2}$ $= 0.423$ and $\omega^2 = \frac{1.2}{0.13}$. $\omega = 3.04$. We find $\omega > b$.

Hence the motion is underdamped.

For critically damped motion we require $\omega = b = 3.04$.

$\therefore D = 2bm = 2\times 3.04\times 0.130 = 0.79$ kg/s.

3. *A damped vibrating body starting from rest has its first amplitude of 0.50 m and it reduces to 0.05 in the same direction after 100 oscillation. Time period of the body as observed is 2.3 s. Find the damping factor and find the first displacement taking correction for damping.*

ANS If the first maximum displacement from the mean position is written as x_1, the maximum displacement in the same direction after 100 oscillations should be x_{201}. We have the relation

$$ln\left(\frac{x_1}{x_n}\right) = (n-1)\lambda. \quad \therefore ln\left(\frac{x_1}{x_{201}}\right) = (201-1)\lambda.$$

$$\therefore \lambda = \frac{1}{200}ln\left(\frac{0.5}{0.05}\right) = \frac{2.3}{200}.$$

Again $\lambda = \frac{bT_d}{2} = \frac{b \times 2.3}{2} = \frac{2.3}{200}$.

$\therefore$ Damping factor, $b = \frac{2}{200} = 0.01$ per sec.

If we take correction for damping the first deflection should be (see text)

$$x = x_1\left(1+\frac{\lambda}{2}\right) = 0.5 \times \left(1+\frac{2.3}{2\times 200}\right) = 0.5028 \text{ m.}$$

4. *A tuning fork of frequency 300 Hz has quality factor 5×10^4. Calculate the time interval after which its energy reduces to 1/10 of its initial energy.*

ANS Quality factor, $Q = \frac{\omega}{2b}$.

$b = \frac{2\pi n}{2Q} = \frac{2\pi \times 300}{2\times 5\times 10^4} = 0.0188$ per sec. If initial energy is E_0, energy reduces to E, given by

$E = E_0 e^{-2bt}$. Given $E = 0.1E_0$. $\therefore e^{2bt} = 10$.

$$\therefore t = \frac{ln 10}{2b} = \frac{ln 10}{2\times 0.0188} = 61.23 \text{ s.}$$

5. *A sound is twice as intense as another. What is the difference in intensity levels of the two?*

ANS Intensity levels of the two are $\beta_1 = 10\log_{10}\left(\frac{I}{I_0}\right)$ dB and $\beta_2 = 10\log_{10}\left(\frac{2I}{I_0}\right)$ dB.

$\therefore$ Difference in intensity level is

$$\beta_2 - \beta_1 = 10[(\log_{10}2 + \log_{10}I - \log_{10}I_0) - (\log_{10}I - \log_{10}I_0)]$$

$= 10\log_{10}2 = 3$ dB (nearly).

6. *Imagine a point source of sound emitting constant sound power. By how many dB does the sound intensity level drop if you move twice as far away from the source?*

ANS As spherical wave emerges from the point source, intensity falls off as the square of distance. Therefore ratio of intensities at distance r and $2r$ is

$$\frac{I_1}{I_2} = \frac{(2r)^2}{r^2} = 4.$$

$\therefore$ Difference in intensity ratio in dB is

$$\beta_2 - \beta_1 = 10\log_{10}\left(\frac{I_2}{I_0}\right) - 10\log_{10}\left(\frac{I_1}{I_0}\right)$$

$$= 10\log_{10}\left(\frac{I_2}{I_1}\right)$$

$$= 10\log_{10}(1/4) = -6.0 \text{ dB}$$

Intensity level decreases by 6 dB.

7. *Intensity level in rock music sometimes reaches 120 dB. Intensity level of ordinary speech is typically 60 dB. Find the ratio of actual intensity of rock music and ordinary speech.*

ANS Difference in intensity level is

$$\beta_R - \beta_S = 10\log_{10}\left(\frac{I_R}{I_0}\right) - 10\log_{10}\left(\frac{I_S}{I_0}\right)$$

$$= 10\log_{10}\left(\frac{I_R}{I_S}\right)$$

$$\therefore 120 - 60 = 60 = 10\log_{10}\left(\frac{I_R}{I_S}\right). \therefore \frac{I_R}{I_s} = 10^6$$

Intensity of rock music can be million times greater than that of ordinary speech.

8. *Intensity level in a noisy room is 100 dB. What is the intensity in watts per cubic metre?*

ANS If I be the intensity, intensity level in dB is

$$\beta = 100 = 10\log_{10}\left(\frac{I}{I_0}\right).$$

$\therefore \frac{I}{I_0} = 10^{10}$. We know $I_0 = 10^{-12}$ watt/m².

$\therefore I = 10^{-2}$ W/m².

9. *There are two independent sound sources which are individually 90 dB and 100 dB. Which is the loudness when they are both sounded together?*

ANS It is certainly not 190 dB. We have the relation for sound level of intensity I in dB unit:

$$\beta = 10\log_{10}\frac{I}{I_0}. \therefore I = I_0 10^{\beta/10}.$$

Therefore intensities of the two sounds are $I_1 = I_0 10^9$ and $I_2 = I_0 10^{10}$. As the two sound waves are incoherent, i.e., neither the two are correlated nor have the same frequency, then the resultant intensity I is the sum of the individual intensities.

$$I = I_1 + I_1 = I_0(1+10^{-1})\times 10^{10} = I_0(1.1\times 10^{10})$$

$\therefore$ Loudness of the resultant sound is

$$\beta = \log_{10}\frac{I}{I_0} = \log_{10}(1.1 \times 10^{10}) = 10.04 \text{ dB.}$$

10. *A sound of frequency 3000 Hz with intensity level of 70 dB produces the same loudness as a standard source of frequency 1000 Hz at a intensity level 67 dB. Find the intensity level of the sound in phons.*

Size of an empty assembly hall is 20 m × 15 m × 10 m. If absorption coefficient is 0.106, calculate reverberation time.

ANS By definition intensity level of the sound is 67 phon.

Reverberation time, $T = \frac{0.167V}{\sum a_i S_i}$. Volume of the room, $V = 20 \times 15 \times 10 = 3000\ m^3$.

Absorption coefficient (a) is same throughout the whole surface.

Total surface area,

$S = 2 \times (20{\times}15{+}20{\times}10{+}15{\times}10) = 1300\ m^2$.

$$T = \frac{0.167 \times 3000}{0.106 \times 1300} = 3.64\ s.$$

QUESTIONS

1. In which condition a body is capable of vibrating? In which condition is vibration called free vibration? How the frequency of such a vibration depends on different factors. What is name of this frequency?
2. When a vibration is called linearly damped vibration? Write down its differential equation of such a vibration, explaining the meanings of the different symbols. Write down its solution when damping is small and explain the nature of the motion .
3. Draw the displacement-time curve of a particle for a linearly damped vibration when damping is small and compare it with that of a free vibration.
4. Explain how damping force influences the nature of vibration of a body.
5. Describe the behavior of a body when it is displaced from its equilibrium position and then released, if it is (i) critically damped and (ii) overdamped. Draw displacement-time curves for these two types of motion.
6. For a linearly damped body show that the energy of vibration decreases at higher rate than its displacement when damping is small. What is time constant of damped system?
7. (i). What is quality factor of a damped system? Which factors it depends upon?

   (ii) 'Lesser quality factor means greater loss in energy per cycle.' Justify the statement.
8. A simple harmonic force begins to act on a body which is capable of vibrating and having small damping. Describe in detail what is observed over a long time.
9. A simple harmonic force begins to act on a body capable of vibrating and having small damping. Write down the differential equation of vibration of the body explaining the meaning of the different terms. Write down the general solution of the equation. Explain what this solution predicts about the nature of vibration of the body.
10. Which factors amplitude of steady forced vibration of a body depends upon?
11. With the help of a suitable curve explain how the amplitude of vibration of a body changes when the frequency of the forcing system increases from a low value and surpasses the natural frequency of vibration of a body.Also show how damping force present does influence during this process?
12. Suppose frequency of the periodic force is exactly equal to the natural frequency of the body. How does the body respond? What is the phase difference between the vibration of the body and the applied periodic force in this condition?
13. What are energy resonance and amplitude resonance and in which conditions these two occur? Which of them is more significant?
14. Calculate the average power supplied by the forcing system during the forced vibration of a body and its maximum value.
15. When resonance of a forced vibration is called sharp and when it is called flat? How damping force affects the sharpness of resonance?
16. Define: half-power frequency, band width and sharpness of resonance of a forced system.
17. Which factors sharpness of resonance of a body undergoing forced vibration depends upon?
18. Derive the relations: $Q = \frac{1}{2b}\sqrt{\frac{k}{m}}$ and $\omega_1\omega_2 = \omega^2$. where Q is called sharpness of resonance and ω is the natural angular frequency of body undergoing forced vibration and others have usual meaning.
19. Why sharp resonance is often a desirable property of a forced system?
20. What are half-power frequencies, band-width and quality factor of a forced system?

21. Prove that the product of two half-power frequencies is equal to the square of the resonant frequency.
22. Explain how different properties of a forced system affect its quality factor?
23. State Fourier's theorem and express it in mathematical terms.
24. What is physical implication of Fourier's theorem?
25. What is Fourier analysis?
26. A square wave is described as function $f(t)$ satisfying the conditions :

    $f(t) = +a$ from $t = 0$ to $t = T/2$.

    $f(t) = -a$ from $t = T/2$ to $t = T$.

    Analyse the function by Fourier's theorem and express in terms of its Fourier's components.
27. A saw tooth wave is shown in Fig. 3.10a Expand the function into a Fourier's series.
28. (i) What is the difference between loudness and intensity of sound?

    (ii) Which factors intensity of sound depends upon?

    (iii) What are the two thresholds of intensity for human ear?

    (iv) Give an approximate idea of the range of intensity of sound over which our ears can hear. Does this range depends on anything else?
29. What are tone and note?
30. What is pitch of a note? Which physical factors it depends upon?
31. What is quality or timbre of a musical sound? How quality of a sound may change without changing its pitch?
32. How loudness depends on intensity of sound? State the Weber-Fechner law. Is this law applicable to perception of sound only?
33. What is intensity level of sound? Define bel and decibel.
34. If intensity level of a sound is 1 bel, why its intensity is ten times the threshold intensity?
35. Intensity level of a sound is 1 dB. How much higher is its intensity above the threshold intensity?
36. What is the whole range of sound intensity le in dB unit?
37. Define phon. Why it is more appropriate tha unit for loudness of sound?
38. The decibel scale is an objective measure sound; the phon scale is more subjective. Ju the statement.
39. Define sone. Why it is more appropriate than p as a unit for loudness of sound?
40. What is the relation by which we may conver to sone and vice versa?
41. In our perception of variation of pitch of no our ears are sensitive to what?
42. Why numerical differences of frequencies are important when we hear music?
43. What is musical interval between two notes? W it is so important? Explain by examples.
44. When interval between the two notes is calle octave? When these are said to be in unison?
45. What are consonance and dissonance in percep of musical sound?
46. When do two notes produce pleasant effect?
47. What is the reason behind the dissonance?
48. What is musical scale?
49. What is diatonic scale? Why it represents absolu harmonic perfection? What is its disadvantage?
50. What is tempered scale? What is its advantage ov diatonic scale?
51. What are acoustic of building concerned with?
52. What is reverberation? Define reverberation tim
53. Explain why reverberation has both disadvantag and advantage.
54. Define absorption coefficient. Define sabin. Wh is regarded as perfect absorber of sound? Ho much a listener a room absorbs sound?
55. State Sabine formula.
56. Give a simple derivation of Sabine formula.
57. Briefly discuss the requirements for good acous in a hall and auditorium and how these a achieved?
58. Briefly discuss how reverberation time is measu

NUMERICAL PROBLEMS

1. A spring-mass oscillator has force constant 4.72 N/m and mass 0.93 kg. What must be the period of a simple harmonic driving force if it is to excite resonance? [*Ans.* 2.79 s]
2. Calculate the change in intensity level when th intensity of sound increases 100 times its origi intensity. [*Ans.* 20 dB]

3. For a particular frequency faintest sound that can be heard and the loudest sound that can be heard without pain have intensities 4.49×10^{-13} W/m^2 and 0.881 W/m^2 respectively. Find their intensity levels in decibel. [*Ans.* – 3.477 dB and 119 dB]

4. Two independent sources of sound have individual loudnesses 50 dB and 55 dB. What would be the loudness when the two are sounded together? [*Ans.* 56.2 dB]

5. An airplane produces sound level of 100 dB at a distance 100 m from it. Find the distance at which the sound will be just audible. You can take the airplane to be a point source of sound. Explain why your result is absurd.
[*Ans.* 10^4 km. Absorption of sound is neglected.]

6. About how many times more intense will the normal ear perceives a sound of 10^{-6} W/m^2 than one of 10^{-9} W/m^2? [*Ans.* Twice as loud]

7. The decibel rating of a hall increases by 10 dB. How much the intensity has increased? [*Ans.* 10 times.]

8. What is the intensity of 60 dB sound? [*Ans.* 10^{-6} W/m^2]

9. Maximum intensity level at a place is fixed at 65 dB. If at another place it is 85 dB, how much greater is the intensity than the fixed maximum value? [*Ans.* 100]

10. The threshold of auditory pain sets in at an intensity level of about $\beta = 140$ dB. What is the corresponding sound wave intensity and sound pressure? Given data: sound pressure and intensity at threshold of hearing are 2×10^{-5} N/m^2 and 10^{-12} watts / m^2.
[*Ans.* 100 W/m^2 and 200 Pa.]

11. A person speaking normally produces a sound level of 40 dB. If it can be assumed that each person speaks at the same level, what is the sound level in a room if 30 people are speaking at the same time ? [*Ans.* 54.77 dB.]
